Perspectives and Problems in Nonlinear Science

A Celebratory Volume in Honour of Larry Sirovich
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Editors: Ehud Kaplan, Jerry Marsden, and Katepalli Sreenivasan
To Larry Sirovich

On the occasion of his 70th birthday,
with much admiration and warmth
from his friends and colleagues worldwide.

Contents

Preface ix
Contributors xi

1 Reading Neural Encodings using Phase Space Methods, by Henry D. I. Abarbanel and Evren Tumer 1

2 Boolean Dynamics with Random Couplings, by Maximino Aldana, Susan Coppersmith and Leo P. Kadanoff 23

3 Oscillatory Binary Fluid Convection in Finite Containers, by Oriol Batiste and Edgar Knobloch 91

4 Solid Flame Waves, by Alvin Bayliss, Bernard J. Matkowsky and Anatoly P. Aldushin 145

5 Globally Coupled Oscillator Networks, by Eric Brown, Philip Holmes and Jeff Moehlis 183

6 Recent Results in the Kinetic Theory of Granular Materials, by Carlo Cercignani 217

7 Variational Multisymplectic Formulations of Nonsmooth Continuum Mechanics, by R. C. Fetecau, J. E. Marsden and M. West 229

8 Geometric Analysis for the Characterization of Nonstationary Time Series, by Michael Kirby and Charles Anderson 263

9 High Conductance Dynamics of the Primary Visual Cortex, by David McLaughlin, Robert Shapley, Michael Shelley and Jacob Wielaard 293


11 A KdV Model for Multi-Modal Internal Wave Propagation in Confined Basins, by Larry G. Redekopp 343

12 A Memory Model for Seasonal Variations of Temperature in Mid-Latitudes, by K. R. Sreenivasan and D. D. Joseph 361

13 Simultaneously Band and Space Limited Functions in Two Dimensions and Receptive Fields of Visual Neurons, by Bruce W. Knight and Jonathan D. Victor 375

14 Pseudochaos, by G. M. Zaslavsky and M. Edelman 421
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Oscillatory Binary Fluid Convection in Finite Containers

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Edgar Knobloch

To Larry Sirovich, on the occasion of his 70th birthday.

ABSTRACT Linear and weakly nonlinear theory of overstable convection in large but bounded containers is reviewed and the results compared with detailed numerical simulations of binary fluid convection in a two-dimensional domain with realistic boundary conditions. For sufficiently negative separation ratios convection sets in as growing oscillations; the corresponding eigenfunctions take the form of ‘chevrons’ of either odd or even parity. These may bifurcate sub- or supercritically. Simulations of $^3$He–$^4$He and water-ethanol mixtures show that the oscillations may equilibrate in finite amplitude chevron states, or that these states are unstable to blinking or repeated transient states. The results compare favorably with available experiments.

Contents

1 Introduction .............................................. 92
2 Abstract Considerations ................................. 93
3 Convection in Binary Mixtures ......................... 100
4 Linear Theory ........................................... 102
  4.1 $^3$He–$^4$He mixtures ................................. 103
  4.2 Water-ethanol mixtures ............................... 108
5 Simulations .................................................. 110
  5.1 $^3$He–$^4$He mixtures in a $\Gamma = 10$ container .... 110
  5.2 Water-ethanol mixtures in $16 \leq \Gamma \leq 17.25$ containers . 122
6 Origin of the Repeated Transients ...................... 128
7 Discussion .................................................... 137
References ...................................................... 141
1 Introduction

Binary fluid mixtures with a negative separation ratio exhibit a wide variety of behavior when heated from below. Particular attention has focused on the transition to various types of traveling waves with increasing Rayleigh number, hereafter $R$. The experimental situation is summarized by Sullivan and Ahlers [1988], Kolodner, Surko, and Williams [1989], Steinberg, Fineberg, Moses, and Rehberg [1989] and Kolodner [1993], and sample data are reproduced in fig. 1.1. These experiments have either been carried out in narrow gap annular containers, or in extended rectangular boxes. The two experimental arrangements differ in a fundamental way. In the former the system is periodic and consequently the initial instability can develop into a uniform pattern of traveling waves. This is no longer so when sidewalls are present: the presence of sidewalls destroys the translation invariance present in the annular (or unbounded) system, with the result that the finite system has only a left-right reflection symmetry. Consequently, the eigenfunctions of the latter system are either odd or even under left-right reflection, but are otherwise unconstrained by the symmetries [Dangelmayr and Knobloch, 1987, 1991; Dangelmayr, Knobloch, and Wegelin, 1991]. In contrast, in the annular (or unbounded) case the presence of translation invariance with periodic boundary conditions forces the eigenfunctions to be sinusoidal functions with a single wavenumber in the horizontal. Such eigenfunctions take the form of left- and right-traveling waves. In systems with up-down reflection symmetry the additional symmetry may also affect the eigenfunctions and constrain the dynamics.

The difference in symmetry between the bounded and unbounded systems is crucial, and is present regardless of the aspect ratio of the system. It suggests that while unbounded systems are best described in terms of amplitude equations for the amplitudes of left- and right-traveling waves, bounded systems should be described in terms of odd and even modes, cf. Landsberg and Knobloch [1996]. As shown by Batiste, Mercader, Net, and Knobloch [1999] these modes typically have a complex spatial structure. We summarize here the properties of these eigenfunctions for the parameter values used in experiments and relate them to two classes of weakly nonlinear theories developed for the onset of oscillatory instability in large aspect ratio domains. In particular we show that for large values of the aspect ratio $\Gamma$ the differences between the growth rates and frequencies of the first two modes that set in both scale as $\Gamma^{-2}$. This result supports the description of the system in terms of an interaction between the first even and odd modes [Landsberg and Knobloch, 1996]. Direct numerical simulations of the governing partial differential equations for both $^3\text{He}-^4\text{He}$ and water-ethanol mixtures confirm the important role played by these pure parity modes, and shed light on the presence of two classes of dynamical behavior observed in the experiments, referred to as blinking states and repeated transients. We show that both of these states are fundamentally
finite-dimensional and that they may occur even in extended systems, provided only that these are not too large in the sense that $1 \ll \Gamma \lesssim |\epsilon|^{-1/2}$, where $\epsilon \equiv (R - R_c)/R_c$ measures the fractional distance from onset of the primary instability. Throughout this article we focus almost exclusively on this regime, since it is amenable to both theory and direct numerical simulation.

![Figure 3](image-url)  
**FIG. 3.** The image intensity measured at a single spatial point in a cell of dimensions $4.90 \times 16.75$ is plotted as a function of time for different values of the Rayleigh number: (a) $\epsilon=0.0006$; (b) $\epsilon=0.0008$; (c) $\epsilon=0.0015$; (d) $\epsilon=0.0030$; (e) $\epsilon=0.0105$; (f) $\epsilon=0.0111$. The repeated-transient states seen at the lowest values of $\epsilon$ [(a), (b)] give way at higher $\epsilon$ to blinking states [(c), (d)] whose period grows quite long at the highest values of $\epsilon$ shown [(e), (f)] and diverges at $\epsilon=0.012$.

**Figure 1.1.** Figure 3 of Kolodner [1993] with $\epsilon \equiv (R - R_c)/R_c$, reproduced with permission.

## 2 Abstract Considerations

We consider first an annular translation-invariant domain with left-right reflection symmetry and length $\Gamma$. Systems of this type necessarily possess a trivial (i.e., $O(2)$-symmetric) basic state, here the conduction state. An oscillatory instability of this state with a finite azimuthal wave number breaks the $O(2)$ symmetry of the system. As a result the multiplicity of the
purely imaginary eigenvalues at criticality is doubled. This is because the reflection-related eigenfunctions \( \exp(\pm ikx) \) (\( k \equiv 2\pi n/\Gamma \neq 0 \)) represent two independent eigenfunctions of the eigenvalue \( i\omega_c \). The dynamics at such a bifurcation (called a Hopf bifurcation) can be described in terms of the (complex) amplitudes of these eigenfunctions, viz.,

\[
\theta(x, z, t) = \text{Re}[v(t) e^{ikx} + w(t) e^{-ikx}] f(z) + \cdots
\]  

(2.1)

Here \( \theta \) represents the departure of the temperature from its conduction profile, \( f(z) \) is the vertical eigenfunction, and the \( \cdots \) represent higher order spatial harmonics. Center manifold theory shows that the latter are slaved to the (slow) dynamics of \( v \) and \( w \). Thus the dynamics near the Hopf bifurcation are entirely determined by the behavior of the two amplitudes \( (v, w) \). The equations for these inherit the equivariance of the physical system under translations \( x \rightarrow x + \ell \) and reflection \( x \rightarrow -x \). An examination of equation (2.1) shows that the translations are equivalent to the operation \( R_\ell : (v, w) \rightarrow (ve^{ik\ell}, we^{-ik\ell}) \), while the reflections are equivalent to \( R_x : (v, w) \rightarrow (w, v) \). In addition, the equations must commute with the operation \( T_\phi : (v, w) \rightarrow e^{i\phi}(v, w) \) representing the effect of time translation, \( t \rightarrow t + \phi/\omega_c \). It follows that the most general equations for \( (v, w) \), truncated at third order, take the form [Knobloch, 1986; Crawford and Knobloch, 1991]

\[
\dot{v} = (\lambda + i\omega)v + b|v|^2v + (a + b)|w|^2v,
\]

\[
\dot{w} = (\lambda + i\omega)w + b|w|^2w + (a + b)|v|^2w.
\]  

(2.2)

In writing these equations we have included the small linear terms \((\lambda + i(\omega - \omega_c))(v, w)\), proportional to the bifurcation parameter \( \epsilon \equiv (R - R_c)/R_c \ll 1 \). In the following we refer to \( v, w \) as the amplitudes of left- and right-traveling waves. The truncated equations are valid under the nondegeneracy conditions \( a_R \neq 0, b_R \neq 0 \) and \( a_R + 2b_R \neq 0 \), where the subscript \( R \) indicates the real part.

Analysis of equations (2.2) shows that at \( \lambda = 0 \) two branches of solutions bifurcate simultaneously from the trivial state \( (v, w) = 0 \). These are the traveling waves \( (v, w) = (v, 0) \) and the standing waves \( (v, w) = (v, v) \). Of course, the symmetries of the problem can be used to generate new solutions from these. Thus the reflection \( R_x \) shows that if \( (v, 0) \) is a solution so is \( (0, v) \) while the translations \( R_\ell \) generate \( (e^{ik\ell}v, e^{-ik\ell}v) \) from \( (v, v) \), i.e., spatial translates of the standing waves. Note that the traveling waves have a spatio-temporal symmetry: \( (v, 0) \) is invariant under \( T_{-k\ell} \circ R_\ell \), a spatial translation followed by an appropriate time translation, while the standing waves form a circle foliated by periodic orbits. Figure 2.1 summarizes the results of the analysis in the \((a_R, b_R)\) plane and shows that at most one of the solutions can be stable, and that this requires that both branches bifurcate supercritically. Moreover, the stable branch is the one with the larger amplitude \( A \equiv \sqrt{|v|^2 + |w|^2} \).
In a laterally bounded domain translation invariance is absent. Thus only the reflection symmetry remains, and this symmetry is a reflection about a particular point, the center \( x = \Gamma/2 \) of the container, rather than about an arbitrary point. The boundaries select from the circle of standing waves (SW) two representatives, with odd or even parity with respect to \( x = \Gamma/2 \), and these bifurcate in succession rather than simultaneously. Moreover, the loss of translation invariance implies that traveling waves (TW) can no longer bifurcate from the trivial state, and must therefore turn into secondary branches. From a symmetry point of view the resulting solutions are periodic in time but lack pure parity, i.e., they are neither odd nor even with respect to \( x = \Gamma/2 \). Consequently there are two such solutions, related by reflection, neither of which has spatio-temporal symmetry. These results follow from group-theoretic considerations [Dangelmayr and Knobloch, 1987, 1991]. But the loss of translation symmetry has other important consequences: it permits complex dynamics. The reason is simple. The two continuous symmetries of equations (2.2), originating in invariance under translations in space and time, imply that the phases \( \text{arg}(v) \), \( \text{arg}(w) \) decouple from the amplitudes \( |v| \), \( |w| \). The equations for the latter are therefore two-dimensional; it is these that lead to fig. 2.1. The loss of translation invariance implies that only one overall phase will decouple, and hence that the dynamics will become three-dimensional, permitting complex dynamics.

These notions can be made explicit [Dangelmayr and Knobloch, 1987, 1991]. To this end we include in equations (2.2) the dominant terms that break translation invariance (i.e., the symmetry \( R_\ell \)) while preserving the reflection and normal form symmetries \( R_x, T_\phi \). The most general such equations with a linear symmetry-breaking term are the equations [Dangelmayr

\[ \begin{align*}
\dot{v} &= (\lambda + i\omega)v + ew + b|v|^2v + (a + b)|w|^2v, \\
\dot{w} &= (\lambda + i\omega)w + ev + b|w|^2w + (a + b)|v|^2w.
\end{align*} \tag{2.3} \]

Here \( e \) is a small but complex coefficient measuring the strength of the symmetry breaking. In writing eqs. (2.3) we have implicitly assumed that small symmetry-breaking cubic terms do not qualitatively affect the resulting bifurcation diagrams (but see Nagata [1991]).

Equations (2.3) have been studied in detail by Dangelmayr and Knobloch [1991], who showed that when \( e \neq 0 \) the double multiplicity Hopf bifurcation splits into two successive simple Hopf bifurcations each of which gives rise to a standing wave. Following Dangelmayr and Knobloch [1987, 1991] we refer to these as \( SW^0 \) and \( SW^\pi \), the prime denoting solutions of equations (2.3) when \( e \neq 0 \). The \( SW^0 \) is an even mode while \( SW^\pi \) is odd. Both solutions are characterized by time-independent amplitudes with \( |v| = |w| \). There are no other primary bifurcations. Analogues of traveling waves, hereafter \( TW' \), exist but only appear through secondary (pitchfork) bifurcations from the standing wave branches. The \( TW' \) are single frequency solutions with \( |v| \neq |w|, |vw| > 0 \), with \( |v| > |w| \) representing a state dominated by left-traveling waves and vice versa. More surprisingly perhaps, depending on the argument of \( e \), an additional secondary Hopf bifurcation can also take place (fig. 2.2). This bifurcation gives rise to a two-frequency state \( MW' \) in which the new frequency describes oscillations in the amplitudes \( |v|, |w| \). Solutions of this type have been seen in experiments on water-ethanol mixtures [Kolodner and Surko, 1988] as well as in numerical simulations [Cross, 1986, 1988; Ning, Harada, and Yahata, 1997; Batiste, Knobloch, Mercader, and Net, 2001a], where they are referred to as blinking states. These states persist only in an interval of \( \lambda \), and in the context of equations (2.3) terminate on the \( TW' \) branch in either a Hopf
or a global bifurcation (fig. 2.2). The appearance of such global bifurcations once translation invariance is broken provides the key to the origin of chaotic dynamics in the perturbed system.

An important additional effect of nonzero $e$ is the appearance of stable behavior in parameter regions where the unperturbed problem has no stable solutions. For example, in region II of fig. 2.1 the SW branch bifurcates forwards and TW backwards, and hence neither is stable. In such a case for $\lambda > 0$ all solutions escape to large amplitude, and a study of the saturation of the instability therefore requires numerical simulations of model equations. This is in contrast to the perturbed problem in which the SW continue to bifurcate supercritically but the first SW branch is initially stable (Clune and Knobloch [1992]: see fig. 2.2b). Consequently the solutions MW and TW which bifurcate from it in secondary bifurcations also enjoy limited parameter windows of stability. Other examples of the effect of the perturbation $e \neq 0$ are described by Hirschberg and Knobloch [1996].

Equations (2.3) arise naturally as the small amplitude limit of the coupled complex Ginzburg-Landau (CCGL) equations that have been used to describe the dynamics of traveling waves in large aspect ratio containers. These equations, defined on the (long) interval $0 \leq x \leq \Gamma/\varepsilon$, take the following form

$$
\begin{align*}
A_T - sA_X &= \Lambda A + \tilde{b}|A|^2A + (\tilde{a} + \tilde{b})|B|^2A + DA_{XX}, \\
B_T + sB_X &= \Lambda B + \tilde{b}|B|^2B + (\tilde{a} + \tilde{b})|A|^2B + DB_{XX},
\end{align*}
$$

(2.4)

together with the model boundary conditions

$$
\begin{align*}
A + \varepsilon(\mu_1A_X + \nu_1B_X) = 0, & \quad B + \varepsilon(\mu_2B_X + \nu_2A_X) = 0, & X &= 0, \\
A - \varepsilon(\mu_2A_X + \nu_2B_X) = 0, & \quad B - \varepsilon(\mu_1B_X + \nu_1A_X) = 0, & X &= \Gamma,
\end{align*}
$$

(2.5)
cf. Cross [1986]. Here $A(X, T), B(X, T)$ denote the envelopes of left- and right-traveling waves with the slow variables $X \equiv \varepsilon x$, $T \equiv \varepsilon^2 t$, and $\tilde{a}, \tilde{b}, \Lambda$ are complex coefficients. The scaled group velocity $s \equiv \frac{1}{\varepsilon} \frac{d\Gamma}{dt}$ is assumed to be of order unity. A straightforward center manifold reduction yields a solution in the form [Dangelmayr, Knobloch, and Wegelin, 1991]

$$
\begin{align*}
\theta(x, t) &= \Re e^{\frac{3}{2}} \left\{ v(T') \exp \left( \frac{s(\Gamma - 2X)}{4D} \right) e^{ik_\infty x} \\
&\quad + w(T') \exp \left( -\frac{s(\Gamma - 2X)}{4D} \right) e^{-ik_\infty x} \right\} e^{i\omega_\infty t} \sin \frac{\pi X}{\Gamma} + O(\varepsilon^{5/2}),
\end{align*}
$$

(2.6)

where $T' \equiv \varepsilon T$, $v(T'), w(T')$ satisfy (2.3), with the frequency $\omega_\infty$ factored out and $\lambda \equiv \Lambda - \Lambda_c$. Here $\Lambda_c = (s^2/4D) + (\pi^2D/\Gamma^2)$, and $\omega_\infty$ and $k_\infty$ are the onset frequency and wavenumber in the unbounded system [Knobloch and Moore, 1988], with $\omega_\infty - \omega_\infty = O(\varepsilon)$. Note that the presence of the boundaries at $X = 0, \Gamma$ shifts the threshold of the instability by an $O(1)$ amount even in the limit $\Gamma \to \infty$. This is because at leading order the boundary conditions (2.5) are absorbing.
The coefficients in (2.3) are computable functions of the parameters appearing in equations (2.4, 2.5). Specifically,

\[ e = -\left(\frac{2\pi^2 D}{T^3}\right)\left[\nu_1 e^{-s\Gamma/2D} + \nu_2 e^{s\Gamma/2D}\right], \]

(2.7)

cf. Dangelmayr, Knobloch, and Wegelin [1991], indicating that in general the perturbation depends exponentially on the aspect ratio. However, since the amplitude of \( e \) can be scaled out the different possible bifurcation diagrams depend only on \( \arg(e) \), a quantity that depends on \( \Gamma \) approximately linearly. Since a change of sign of \( e \) is equivalent to a change of sign of either \( v \) or \( w \) we see that increasing \( \arg(e) \) by \( \pi \) is equivalent to replacing an even mode by an odd mode or vice versa. Equation (2.7) suggests therefore that the dynamics depend sensitively on \( \Gamma \) for \( n\pi < \arg(e) < (n+1)\pi \), but more or less repeat for \( (n+1)\pi < \arg(e) < (n+2)\pi \) except for a change in the parity of the primary modes SW'. This prediction is confirmed by both experiments and direct numerical simulations, as discussed further in section 5.

A striking consequence of the nonzero group velocity is the fact that the two constituent eigenfunctions peak in opposite halves of the container. As a result the waves near one end of the container travel in the opposite direction to those at the other end. Patterns obtained by the substitution of the simple solutions SW', TW' and MW' for \( v \) and \( w \) in the wavefunction (2.6) resemble, respectively, the ‘chevron’, confined state, and blinking patterns observed in the experiments [Kolodner, Surko, and Williams, 1989; Steinberg, Fineberg, Moses, and Rehberg, 1989; Andereck, Liu, and Swinney, 1986; Croquette and Williams, 1989]. In fact equations (2.3) have chaotic solutions as well [Hirschberg and Knobloch, 1996] and these may be responsible for the irregularly reversing waves observed both in binary fluid convection [Kolodner, Surko, and Williams, 1989; Steinberg, Fineberg, Moses, and Rehberg, 1989] and in the counter-rotating Taylor-Couette system [Andereck, Liu, and Swinney, 1986]. These are typically the consequence of a global bifurcation. For example, depending on \( \arg(e) \) or equivalently on the aspect ratio \( \Gamma \), the heteroclinic connection with which the MW' branch terminates in region III (see fig. 2.2a and Knobloch [1996]) may connect two saddle-foci (the TW') with eigenvalues \( \lambda_u, -\lambda_s \pm i\Omega, \lambda_u > 0, \lambda_s > 0, \Omega > 0 \), satisfying the Šil'nikov inequality \( 0 < \delta < 1 \). Here \( \delta \equiv \lambda_s/\lambda_u \). The formation of a global connection of this type thus results in Šil'nikov dynamics [Glendinning and Sparrow, 1984], and so provides a natural mechanism for the origin of chaotic reversals.

While promising, a comparison of the above approach with the CCGL equations suggests that the distant boundaries may only be treated perturbatively when propagative effects are weak and \( \varepsilon \) is finite. However, this requirement prevents us from proceeding to the asymptotic limit \( \varepsilon \to 0 \) required by the theory. Consequently we develop below a theory based on the even and odd modes characteristic of a bounded container. We let
(z_+, z_-) be the (complex) amplitudes of these modes, and note that due to the parabolic nature of the neutral stability curve in unbounded systems the first even and odd modes will typically come in in close succession, well separated from the next mode, cf. Jacqmin and Heminger [1994]. If this is so it makes sense to project the partial differential equations onto these two modes. The resulting equations must be equivariant with respect to

\[ R_x : (z_+, z_-) \rightarrow (z_+, -z_-), \]  

owing to the reflection symmetry \( x \rightarrow \Gamma - x \). However, in the large aspect ratio limit one must expect that there is in addition an interchange symmetry between the odd and even modes since these are effectively indistinguishable throughout most of the domain. Thus the normal form should also be equivariant under [Landsberg and Knobloch, 1996]

\[ I : (z_+, z_-) \rightarrow (z_-, z_+). \]  

These two reflections generate the symmetry \( D_4 \). However, since the interchange symmetry is not exact for any finite \( \Gamma \) (at any finite \( \Gamma \) the first mode is either odd or even, except for a discrete set of \( \Gamma \)) this \( D_4 \) symmetry is broken. The dominant interchange-breaking terms are once again linear, and hence the system should be described by amplitude equations of the form [Landsberg and Knobloch, 1996]

\begin{align}
\dot{z}_+ &= (\mu_+ + i\omega_+)z_+ + A|z_+|^2z_+ + B|z_-|^2z_+ + C\bar{z}_+z_-^2, \\
\dot{z}_- &= (\mu_- + i\omega_-)z_- + A|z_-|^2z_- + B|z_+|^2z_- + C\bar{z}_-z_+^2. \tag{2.10}
\end{align}

Here \( A, B, C \) are complex \( O(1) \) coefficients and the small interchange-breaking parameters \( \Delta |\mu| \equiv \mu_+ - \mu_- \), \( \Delta \omega \equiv \omega_+ - \omega_- \) capture the effects of a finite aspect ratio. As shown below both \( \Delta \mu \) and \( \Delta \omega \) are \( O(\Gamma^{-2}) \) for large \( \Gamma \); consequently the two modes remain close to 1:1 resonance for all \( \Gamma \) and the resonant terms \((\bar{z}_+z_-^2, \bar{z}_-z_+^2)\) must be retained.

It is of interest to rewrite equations (2.10) in terms of the traveling wave coordinates \((v, w)\), where \(z_+ = v + w, \ z_- = v - w\):

\begin{align}
\dot{v} &= (\mu + i\omega)v + \frac{1}{2}(\Delta \mu + i\Delta \omega)w + b|v|^2v + (a + b)|v|^2w + cw^2\bar{v}, \\
\dot{w} &= (\mu + i\omega)w + \frac{1}{2}(\Delta \mu + i\Delta \omega)v + b|w|^2w + (a + b)|v|^2w + cw^2\bar{w}, \tag{2.11}
\end{align}

where \( \mu \equiv \frac{1}{2}(\mu_+ + \mu_-) \), \( \omega = \frac{1}{2}(\omega_+ + \omega_-) \), and \( a = A - B - 3C \), \( b = A + B + C \), \( c = A - B + C \). There is a notable difference between these equations and (2.3): the presence of the terms \((w^2\bar{v}, v^2\bar{w})\) is evidently nonperturbative and indicates that the sidewalls play an important role in the near-onset behavior of the system, however large the system! Indeed, as demonstrated by Renardy [1999], equations of the form (2.11), and hence (2.10), can
also be derived via center manifold reduction of a pair of (local) coupled complex Ginzburg-Landau equations (2.4) on a periodic domain with the (generic) boundary conditions

\begin{align}
\mathcal{A}(0, t) + iB(0, t) &= \mathcal{A}(\Gamma, t) - iB(\Gamma, t) = 0, \\
\mathcal{A}X(0, t) - iBX(0, t) &= \mathcal{A}X(\Gamma, t) + iBX(\Gamma, t) = 0.
\end{align}

(2.12)

Since the two approaches just summarized lead to qualitatively different predictions in the large aspect ratio limit, we focus in the following on some of the assumptions behind them, and confront the predictions with direct numerical simulations in two dimensions.

In the same spirit one can examine the effect of breaking of translation invariance on the transition between oscillatory and steady convection that occurs when \( \omega_c \) is small, i.e., \( S \approx S_{TB} \). In the absence of sidewalls the different possibilities are captured by the normal form for the Takens-Bogdanov bifurcation with \( O(2) \) symmetry [Knobloch, 1986]. Breaking this symmetry at leading order while preserving reflection symmetry leads to the equation

\begin{equation}
\ddot{\theta} = \mu \theta + \nu \dot{\theta} + A |\theta|^2 \theta + C(\dot{\theta} + \dot{\theta}) + D|\theta|^2 \ddot{\theta} + E\dddot{\theta} + F\dddot{\theta}.
\end{equation}

(2.13)

Here \( \mu, \nu \) are two real unfolding parameters, and we have written \( \theta(x, z, t) = \text{Re} v(t) \exp ikx + \cdots \). The coefficients \( A, \ldots, F \) are also real, with \( E, F \) breaking the translation symmetry \( v \rightarrow v \exp ik \ell \). Unfortunately the properties of this equation for \( EF \neq 0 \) remain unknown.

### 3 Convection in Binary Mixtures

Binary fluid mixtures are characterized by the presence of cross-diffusion terms in the diffusion matrix. In liquids the dominant cross-diffusion term is the Soret term, and the sign of this term determines the behavior of the mixture in response to an applied temperature gradient. For mixtures with a negative Soret coefficient the heavier component migrates towards the hotter boundary, i.e., a concentration gradient is set up that opposes the destabilizing temperature gradient that produced it. Under these conditions the onset of convection may take the form of growing oscillations. This is the situation that is of interest here.

We consider a binary mixture in a two-dimensional rectangular container \( D \equiv \{ x, z \mid 0 \leq x \leq \Gamma, -\frac{1}{2} \leq z \leq \frac{1}{2} \} \) heated uniformly from below, and nondimensionalize the equations using the depth of the layer as the unit of length and \( t_d \), the thermal diffusion time in the vertical, as the unit of time. In the Boussinesq approximation appropriate to the experiments the
resulting equations take the form [Clune and Knobloch, 1992]

\[
\begin{align*}
\dot{u} + (u \cdot \nabla)u &= -\nabla P + \sigma R[\theta(1 + S) - S\eta]\hat{z} + \sigma \nabla^2 u, \\
\dot{\theta} + (u \cdot \nabla)\theta &= w + \nabla^2 \theta, \\
\dot{\eta} + (u \cdot \nabla)\eta &= \tau\nabla^2 \eta + \nabla^2 \theta, \\
\end{align*}
\]

(3.1)

(3.2)

(3.3)

together with the incompressibility condition

\[
\nabla \cdot u = 0.
\]

(3.4)

Here \(u \equiv (u, w)\) is the velocity field in \((x, z)\) coordinates, \(P\) is the pressure, and \(\theta\) denotes the departure of the temperature from its conduction profile, in units of the imposed temperature difference \(\Delta T\). The variable \(\eta\) is defined such that its gradient represents the dimensionless mass flux. Thus \(\eta \equiv \theta - C\), where \(C\) denotes the concentration of the heavier component relative to its conduction profile in units of the concentration difference that develops across the layer as a result of the Soret effect. The system is specified by four dimensionless parameters: the Rayleigh number \(R\) providing a dimensionless measure of the imposed temperature difference \(\Delta T\), the separation ratio \(S\) that measures the resulting concentration contribution to the buoyancy force due to the Soret effect, and the Prandtl and Lewis numbers \(\sigma, \tau\), in addition to the aspect ratio \(\Gamma\).

To model the experiments we take the boundaries to be no-slip everywhere, with the temperature fixed at the top and bottom and no sideways heat flux. The final set of boundary conditions is provided by the requirement that there is no mass flux through any of the boundaries. The boundary conditions are thus

\[
\begin{align*}
u = n \cdot \nabla \eta &= 0 \text{ on } \partial D, \\
\theta &= 0 \text{ at } z = \pm 1/2, \\
\theta_x &= 0 \text{ at } x = 0, \Gamma.
\end{align*}
\]

(3.5)

(3.6)

Here \(\partial D\) denotes the boundary of \(D\).

Equations (3.1-3.6) are equivariant with respect to the operations

\[
R_x : \quad (x, z) \to (\Gamma - x, z), \quad (\psi, \theta, C) \to (-\psi, \theta, C),
\]

(3.7)

\[
\kappa : \quad (x, z) \to (x, -z), \quad (\psi, \theta, C) \to (-\psi, -\theta, -C),
\]

(3.8)

where \(\psi(x, z, t)\) is the streamfunction, defined by \((u, w) = (-\psi_z, \psi_x)\). These two operations generate the symmetry group \(D_2\) of a rectangle. It follows that even solutions, i.e., solutions invariant under \(R_x\), satisfy \((\psi(x, z), \theta(x, z), C(x, z)) = (-\psi(\Gamma - x, z), \theta(\Gamma - x, z), C(\Gamma - x, z))\) at each instant in time, while odd solutions are invariant under \(\kappa R_x\) and satisfy \((\psi(x, z), \theta(x, z), C(x, z)) = (\psi(\Gamma - x, -z), -\theta(\Gamma - x, -z), -C(\Gamma - x, -z))\), again at each instant of time. At midlevel, \(z = 0\), these solutions therefore satisfy

\[
(\psi(x, 0), \theta(x, 0), C(x, 0)) = (\psi(\Gamma - x, 0), -\theta(\Gamma - x, 0), -C(\Gamma - x, 0)).
\]
4 Linear Theory

To determine the critical value of the Rayleigh number $R$ at which the conduction state $\psi = \theta = C = 0$ loses stability to overstable convection and the corresponding frequency $\omega_c$, we look for solutions to the linearized equations of the form $f(x, z)e^{(s + i\omega)t}$. The condition $s = 0$ defines the onset of instability and yields a complex condition that can be solved for $R \equiv R_c$ and $\omega \equiv \omega_c$ as a function of the aspect ratio $\Gamma$ for various values of $\sigma$, $\tau$, and the separation ratio $S$. In the following we summarize our results for the parameters $\sigma = 0.6$, $\tau = 0.03$ (typical of $^3$He-$^4$He mixtures) and $\sigma = 6.97$, $\tau = 0.0077$ (typical of water-ethanol mixtures). These choices are motivated by the experiments of Sullivan and Ahlers [1988] and Kolodner [1993], respectively.

The solution of this problem [Batiste, Mercader, Net, and Knobloch, 1999] indicates that the competition between odd and even modes in this system takes one of two basic forms, depending on the separation and aspect ratios. When $|S|$ is small (i.e., close to $|S_{TB}|$, the Takens-Bogdanov point) and $\Gamma$ not too large the mode interaction takes the form familiar from Rayleigh-Bénard convection with non-Neumann boundary conditions: the neutral curves $R_c(\Gamma)$ divide neatly between different families with no intermingling among them. Each family consists of a pair of braided neutral curves, one for an odd mode and the other for an even mode, with each

Figure 4.1. The $(\Gamma, S)$ plane for $^3$He-$^4$He parameters showing the approximate location of mode avoidance (hatched region) and of mode crossing (unhatched region).
family well separated from the next, at least for the low-lying families. The crossings between odd and even modes within each family are structurally stable because of their different parity. For the case of interest here, i.e., \( \Gamma \) and \( |S| \) large enough, the situation is quite different. There are now no distinct families of neutral curves and all modes (including like-parity modes) cross. In general these mode crossings are all structurally stable, either because the modes have opposite parity, or because their frequencies at the mode crossing are nonresonant. Figure 4.1 shows the location of the transition between these types of behavior in the \((\Gamma, S)\) plane.

### 4.1 \(^3\)He–\(^4\)He mixtures

[Figure 4.2](#) Onset of convection in \(^3\)He–\(^4\)He mixtures (\( \sigma = 0.6, \tau = 0.03 \)) in moderate aspect ratio containers. (a) The critical Rayleigh number \( R_c \) and (b) the corresponding frequency \( \omega_c \) for the first even (solid line) and odd (dashed line) mode as a function of the aspect ratio \( \Gamma \) for \( S = -0.001 \). (c,d) The same but for \( S = -0.01 \).

We describe first the results for typical \(^3\)He–\(^4\)He parameters, \( \sigma = 0.6, \tau = 0.03 \), and modest aspect ratios, \( 4 \leq \Gamma \leq 12 \). Figure 4.2 shows the eigenvalues \( R_c(\Gamma) \) and \( \omega_c(\Gamma) \) for two values of the separation ratio, \( S = -0.001 \) and \( S = -0.01 \), in each case for the first even (solid line) and the first odd (dashed line) mode. For \( S = -0.001 \) fig. 4.2 reveals an oscillatory approach of both sets of curves towards the critical Rayleigh number \( R_\infty \) and \( \omega_\infty \) for an unbounded domain. The braiding of the neutral stability curves \( R(\Gamma) \) seen in fig. 4.2a is familiar from studies of stationary Rayleigh-Bénard convection [Daniels, 1977; Nagata, 1990; Hirschberg and Knobloch, 1997]. Because of the braiding the neutral stability curves for the first odd and
even modes cross repeatedly. Such mode crossings indicate the presence of codimension two bifurcations. Figure 4.2b shows that at these points the frequencies of the competing modes are distinct. Consequently these mode interaction points correspond to nonresonant double Hopf bifurcations. However, when \( S = -0.01 \) the situation changes: for large enough aspect ratios (\( \Gamma > 10 \)) the two neutral curves develop cusps (fig. 4.2c). The presence of these cusps is reflected in the discontinuous jumps in the corresponding frequency curves (fig. 4.2d). Figure 4.3 shows the development of these cusps with decreasing \( S \), focusing on the range \( 10 \leq \Gamma \leq 12 \). The figure shows the first two even (solid lines) and odd (dashed lines) modes at (a,b) \( S = -0.005 \), (c,d) \( S = -0.008 \) and (e,f) \( S = -0.01 \). In fig. 4.3a thick (thin) lines are used to indicate the first (second) mode of each parity and this coding is used to identify the corresponding modes in figs. 4.3c,e. In the latter the dotted and dashed-dotted curves indicate even and odd modes originating from yet higher modes in fig. 4.3a. Observe that in fig. 4.3a the neutral stability curves for the two odd modes avoid one another, as do the corresponding curves for the two even modes. At the same time the two sets of frequency curves intertwine. As \( S \) decreases the two odd modes come together near \( \Gamma = 10.5 \) and their frequencies coalesce, apparently with the frequency of the primary even mode; similar behavior involving the two even modes takes place near \( \Gamma = 11.5 \) (see fig. 4.3d). At somewhat smaller \( S \) the two Rayleigh number curves cross transversely (fig. 4.3e) as the first and second modes of each parity trade place, forming the cusps seen in fig. 4.2c. In the range of aspect ratios shown this happens first for the even modes, closely followed by the odd modes. At the same time the corresponding frequency curves separate and thereafter no longer cross. The same interchange mechanism is also responsible for the appearance of the cusp in the second even mode neutral stability curve near \( \Gamma = 10.3 \) with a yet higher order even mode involved (dotted line), with similar behavior occurring for the second odd mode near \( \Gamma = 11.3 \) as well (fig. 4.3c). These results suggest (and more detailed calculations [Batiste, Mercader, Net, and Knobloch, 1999] confirm) that the necessary crossings between modes of like parity originating in adjacent families are mediated by double Hopf bifurcations with 1:1 resonance located at discrete points \( (R_c^{(3)}, \Gamma_c^{(3)}, S_c^{(3)}) \) in the three-dimensional parameter space \( (R, \Gamma, S) \) (cf. fig. 4.1).

In fig. 4.4 we show the bifurcating modes for \( \Gamma = 10 \) when \( S = -0.001 \) (top panels) and \( S = -0.01 \) (lower panels). Both modes are even and are represented in the form of space-time diagrams, with time increasing upwards. When \( S = -0.001 \) the eigenvector takes the form of a standing wave, with the dynamics at the two sidewalls in phase. The amplitude of the standing oscillations peaks in the middle of the container and decreases towards the sidewalls. There is a considerable phase lag between the temperature and concentration oscillation, a consequence of the small value of \( \tau \). As \( S \) decreases the eigenvector gradually develops into a ‘chevron’-like
Figure 4.3. Details of the reconnection process between two modes of like parity for (a,b) $S = -0.005$, (c,d) $S = -0.008$, and (e,f) $S = -0.01$ when $\sigma = 0.6$, $\tau = 0.03$. In (a) solid (dashed) lines denote even (odd) modes while thick (thin) lines denote first (second) modes of each type.

Figure 4.4. The eigenfunction $(\psi, \theta, C)$ of the linear stability problem at $z = 0$ for $\Gamma = 10$, $\sigma = 0.6$, $\tau = 0.03$, $S = -0.001$, $R_c = 1784.09$, $\omega_c = 0.267$ (left panels), and $S = -0.01$, $R_c = 1827.72$, $\omega_c = 1.35$ (right panels), shown in the form of space-time diagrams with $0 \leq x \leq \Gamma$ horizontally and time increasing upwards. The solutions are sinusoidal with period $T \equiv 2\pi/\omega_c$. In the former case the eigenfunctions are standing waves but become ‘chevron’-like in the latter. The amplitude variation across the domain is indicated below each space-time plot.
state, consisting of waves propagating outward from the container center in such a way that the reflection symmetry in \( x = \Gamma / 2 \) is preserved at all times (fig. 4.4b). In contrast an odd parity chevron (at \( z = 0 \)) is at all times odd with respect to this reflection, cf. fig. 4.8b. Note that despite appearances these solutions are strictly sinusoidal in time: the periodic defect formation at \( x = \Gamma / 2 \) arises because the eigenfunction \( \psi \) is a superposition of four functions each of which has the form \( \psi_j \exp \left( i(\omega_c t \pm k_j x) \right) \), \( j = 1, \ldots, 4 \), with the \( k_j \) possibly complex. Each of these functions describes waves propagating with (local) phase velocity \( \pm \omega_c / \text{Re} k_j \). In the bulk the eigenfunction is dominated by the largest contribution; this contribution has a real wavenumber and describes oscillations that are almost standing. However, when the time-dependent amplitude of this component passes through zero (which occurs twice per period) the remaining contributions briefly reveal themselves. In the eigenfunction shown the largest of these has a relatively large phase velocity, and is responsible for the episodic propagation that is so characteristic of the eigenfunctions shown. In both fig. 4.4a,b the dominant local wavenumber remains remarkably uniform across the cell despite the nonuniformity of the amplitude of the eigenfunction. The sources (or sinks) described by these linear eigenfunctions persist into the nonlinear regime (section 5), indicating that in the present system these defects have a nontopological origin.

It is of interest to compare the above results with those for \( \Gamma = 34 \), the aspect ratio used by Sullivan and Ahlers [1988]. In fig. 4.5 we show the neutral stability curves and corresponding frequencies for the first four modes in the range \( 33 \leq \Gamma \leq 35 \) for \( S = -0.001 \), \( S = -0.004 \) and \( S = -0.01 \). A comparison with figs. 4.2 and 4.5 reveals that the frequencies of the dominant modes are determined primarily by the fluid parameters and not the aspect ratio. This is because the oscillations are bulk oscillations that are modified but not caused by the presence of sidewalls. Figure 4.5a shows that when \( |S| \) is sufficiently small the first two families of neutral curves are separated by a gap that is much larger than the amplitude of the braids within each family. This is typical of what happens in Rayleigh-Bénard convection in large domains with non-Neumann boundary conditions [Hirschberg and Knobloch, 1997]. However, in the case of overstability this behavior changes as \( |S| \) increases (fig. 4.5c) and begins to look like that shown in fig. 4.5e. This figure shows the corresponding neutral curves for \( S = -0.01 \) and reveals the crossing of adjacent even modes. This mode crossing involves a nonresonant double Hopf bifurcation (fig. 4.5f) and is the result of a resonant 1:1 mode crossing at \( S = -0.00403 \) (see figs. 4.5c,d), i.e., it is formed by the same process as that leading to the nonresonant crossings shown in figs. 4.3e,f. The fact that the frequency curves in fig. 4.5f are essentially parallel “straight lines” confirms that this mode crossing is “far” from the 1:1 resonance at \( S_c^{(3)} = -0.00403 \). The figure is also in agreement with the plausible hypothesis that in large aspect ratio systems the frequencies of
the first few modes must take the form
\[ \omega_n \sim \omega_{\infty} + c_{1n} \Gamma^{-1} + c_{2n} \Gamma^{-2} + \cdots, \quad n = 1, 2, \ldots, \quad (4.1) \]
where the index \( n \) specifies the order in which the modes become primary as \( \Gamma \) increases. Thus the \( n \)th mode is primary in the interval \( \Gamma_{n-1} \leq \Gamma \leq \Gamma_n \), etc. It follows that \( n = \mathcal{O}(\Gamma) \). The results of fig. 4.5 suggest that \( c_{1n} \sim c_1, \)
\( c_{2n} \sim nc_2, \) where \( c_1 \) and \( c_2 \) are \( \mathcal{O}(1) \) constants independent of \( n \) and \( \Gamma \). Then at any (large) \( \Gamma \) there is an \( \mathcal{O}(\Gamma^{-1}) \) correction to \( \omega_{\infty} \), while \( \Delta_n \omega = \omega_{n+1} - \omega_n = \mathcal{O}(\Gamma^{-2}) \). Thus near any particular \( \Gamma \) the quantity \( \Delta_n \omega \) takes the form, as a function of \( \Gamma \), of a set of equally spaced almost horizontal lines. We have checked that similar behavior occurs for \( S = -0.021 \) as well. Figure 4.5 therefore suggests that for large \( \Gamma \) the splitting \( \Delta \omega \) in frequency and \( \Delta R \) in Rayleigh number between the first odd and even modes are both of the same order as \( \Gamma \to \infty \). This result supports the hypothesis [Landsberg and Knobloch, 1996] that the normal form describing the interaction between odd and even modes in the large aspect ratio limit has approximate \( D_4 \) symmetry, as discussed in section 2.

Figure 4.6 shows the first odd temperature eigenfunctions for \( S = -0.001 \) and \( S = -0.021 \), again in the form of space-time diagrams. In the former
Figure 4.6. Odd parity eigenfunctions \( \theta(x,0,t) \) of the linear stability problem for \( \Gamma = 34, \sigma = 0.6, \tau = 0.03, S = -0.001, R_e = 1768.12, \omega_c = 0.272 \) (left panel), and \( S = -0.021, R_e = 1833.86, \omega_c = 2.025 \) (right panel), shown in the form of space-time diagrams with \( 0 \leq x \leq \Gamma \) horizontally and time increasing upwards. The solutions are sinusoidal with period \( T \equiv 2\pi/\omega_c \). The eigenfunctions in the former case are standing waves but become 'chevron'-like in the latter. The lower panels show the corresponding midplane temperature profiles at \( t = 0.3T \), \( t = 0.5T \), respectively.

In the case the eigenfunction is essentially a standing oscillation with a wavelength that is once again very uniform across the container despite the fact that the amplitude varies substantially (lower left panel). When \( S = -0.021 \) the direction of propagation is outwards with the center of the container having become a source. The amplitude now has a local minimum at the center and increases outwards, peaking near the sidewalls (lower right panel), cf. eq. (2.6) with \( s > 0 \). For aspect ratios this large the odd and even (not shown) eigenfunctions are essentially indistinguishable [Batiste, Mercader, Net, and Knobloch, 1999].

4.2 Water–ethanol mixtures

In fig. 4.7 we show the linear theory results for \( S = -0.021, \sigma = 6.97, \tau = 0.0077 \), corresponding to the experimental mixture used by Kolodner [1993]. The corresponding critical eigenfunctions are shown in fig. 4.8 in the form of space-time diagrams for the three fields \( (\psi, \theta, C) \) evaluated at \( z = 0 \). As shown in fig. 4.7a the first odd and even modes cross near \( \Gamma = 16.8 \). Unlike the odd-odd crossing near \( \Gamma = 16.25 \) this mode interaction
is accessible from the conduction state and hence plays an important role in the dynamics of the system (see below).

Figure 4.7. (a) The critical Rayleigh number $R_c$ and (b) the corresponding frequency $\omega_c$ for $S = -0.021$, $\sigma = 6.97$, $\tau = 0.0077$ as a function of the aspect ratio $\Gamma$. Solid (broken) lines indicate even (odd) parity chevrons. The solid dots correspond to the solutions shown in fig. 5.20.

Figure 4.8. The eigenfunction $(\psi, \theta, C)$ of the linear stability problem at $z = 0$ for $\Gamma = 16.25$, $S = -0.021$, $\sigma = 6.97$, $\tau = 0.0077$ and (a) $R_c = 1776.30$, $\omega_c = 2.819$ (even chevron), (b) $R_c = 1787.47$, $\omega_c = 2.686$ (odd chevron), shown in the form of space-time diagrams with $0 \leq x \leq \Gamma$ horizontally and time increasing upwards. The solutions are sinusoidal with period $2\pi/\omega_c$. 
5 Simulations

Direct numerical simulations of equations of the form (3.1-3.4) in two dimensions with idealized boundary conditions have revealed the presence of complex dynamics associated with standing waves in small aspect ratio systems [Knobloch, Moore, Toomre, and Weiss, 1986] and of chevron-like states in larger domains [Deane, Knobloch, and Toomre, 1988], both of which are related to finite-dimensional dynamics [Knobloch, Proctor, and Weiss, 1993]. Three-dimensional simulations of Rayleigh-Bénard convection in order one domains likewise yield evidence for finite-dimensional behavior [Sirovich and Deane, 1991]. Unfortunately for the parameter values and boundary conditions relevant to experiments equations (3.1-3.6) possess very long transients, even in two dimensions, requiring considerable patience in order to obtain reliable results. We solve these equations using a time-splitting method with an improved boundary condition for the pressure and a second order accurate time integration scheme based on a modified Adams-Bashforth formula [Hugues and Randriamampianina, 1998]. For the spatial discretization we use a Chebyshev collocation pseudospectral method [Zhao and Yedlin, 1994]. In all cases the time step and the number of collocation points used was adjusted until the solutions converged. Typically we used 170 collocation points in the $x$-direction and 30 collocation points in the $z$-direction, with a time step of $10^{-3} t_d$.

Throughout we use the vertical velocity at the points $(x,z) = (0.13 \Gamma, 0)$ (near the left sidewall) and $(0.87 \Gamma, 0)$ (near the right sidewall) as a proxy for shadowgraph intensity measurements (see fig. 1.1). Moreover, monitoring point quantities at mirror locations suffices to determine the spatial symmetry properties of the various possible time-dependent states.

5.1 $^3$He–$^4$He mixtures in a $\Gamma = 10$ container

For $S = -0.001$ simulations of the growing instability at $R = 1785 > R_c \approx 1784.088$ show that the instability saturates in an even parity standing wave with frequency $\omega_1 = 0.25$ near the critical frequency $\omega_c \approx 0.2675$. Figure 5.1 shows the time series of the saturated vertical velocity $w(x = 0.87 \Gamma, z = 0, t)$, and indicates that the primary bifurcation is a supercritical Hopf bifurcation; no evidence of hysteresis was found. With increasing Rayleigh number this state undergoes a (subcritical) Hopf bifurcation that introduces a new frequency $\omega_2$ into the dynamics. Strictly speaking this bifurcation is a torus bifurcation. However, in the following we do not distinguish between Hopf bifurcations of equilibria and of periodic orbits (or tori), since resonance phenomena appear to play little role in the observed dynamics. Figure 5.1 shows that stable single frequency and two-frequency states coexist at $R = 1785.5$.

The two-frequency state can be identified with the blinking states predicted by the abstract theory. Figure 5.2 shows that this state has the
3. Oscillatory Binary Fluid Convection

Figure 5.1. Time series $w(x = 0.87\Gamma, z = 0, t)$ in a $^3\text{He-}^4\text{He}$ mixture with $S = -0.001$, $\sigma = 0.6$, $\tau = 0.03$ for several different values of the Rayleigh number.

Figure 5.2. As for fig. 5.1 but showing a symmetric periodic blinking state at $R = 1786$. (a,b) Contours of $\theta(x, z, t)$ and $C(x, z, t)$ at $t = 4000$. (c,d) $w(x = 0.87\Gamma, z = 0, t)$ and $w(x = 0.13\Gamma, z = 0, t)$.

required symmetry: if we ignore for the moment the fast frequency $\omega_1$ the blinking state has the symmetry $R_x w(x, 0, t) = w(x, 0, t + T_2/2)$, where $T_2 \equiv 2\pi/\omega_2$ is the blinking period. In the following we refer to states of this type as symmetric periodic blinking states. When $R$ is increased to $R = 1786.2$ this state loses stability and the system jumps to a large amplitude even parity steady state (fig. 5.3). The modulation period $T_2$ appears
Figure 5.3. The large amplitude steady state reached when $R$ is increased from $R = 1786$ to $R = 1786.2$.

Figure 5.4. (a) The blinking period $T_2$ in units of the Hopf period $T_0 \equiv 2\pi/\omega_c$ as a function of $R$ near the transition to steady convection. (b) The corresponding Nusselt number.

to diverge logarithmically as this transition is approached (fig. 5.4), suggesting that the oscillations disappear when the two-torus collides with an (unstable) steady state branch. A fit to the theoretical prediction

$$T_2 = -2\lambda_u^{-1} \ln |R - R_h| + d$$

leads to the estimates $R_h \approx 1786.112$, $\lambda_u \approx 0.2452$, $d \approx 8.1537$. Here $\lambda_u$ is to be identified with the leading unstable eigenvalue of the steady state.

Figure 5.5 shows that when $S = -0.021$ the transition to steady convection is quite different. Although the primary instability is still to an even mode ($R_c = 1855.75$, $\omega_c = 2.076$) the bifurcation is now slightly hysteretic, so that stable even chevrons are present even for $R < R_c$. As $R$ is raised a second frequency appears in the time series, corresponding to the onset of a symmetric blinking state. This bifurcation is apparently also subcritical. This behavior is qualitatively similar to that described for $S = -0.001$. However, with increasing $R$ the modulation period $T_2$ begins to increase but then apparently saturates at a finite value (fig. 5.5d). At the same time the period $T_1 \equiv 2\pi/\omega_1$ appears to diverge (fig. 5.5c). These properties are reflected in the time series presented in fig. 5.5a, which show that at $R = 1862.0$ the blinking state loses stability to steady convection from the lowest frequency portion of the wavetrain, and indicate that the transition to steady convection now occurs via a radically different mechanism (see...
Figure 5.5. A $^3$He-$^4$He mixture with $S = -0.021$, $\sigma = 0.6$, $\tau = 0.03$ in a $\Gamma = 10$ container. (a) Time series $w(x = 0.87\Gamma, z = 0, t)$ for different values of $R$ increasing upwards ($1859.5 < R < 1862.0$), showing a hysteretic transition to steady convection at $R \approx 1862.0$. (b) The corresponding Nusselt number as a function of $R$. (c) The chevron period $T_1 \equiv 2\pi/\omega_1$ as a function of $R$. (d) The blinking period $T_2 \equiv 2\pi/\omega_2$ as a function of $R$. The state at $R = 1861.8$ resembles the “fish state” observed in experiments.

below). In the following we refer to the state just prior to this transition as a fish state, cf. Kolodner and Surko [1988].

The case $S = -0.1$ is even more interesting. Here $R_c = 1972.13$, $\omega_c = 4.918$. In this case the primary bifurcation to the (even) chevron state is substantially subcritical, and the first stable nonlinear state takes the form of a repeated transient (fig. 5.6). States of this type were studied in detail by Kolodner [1993] in experiments on water-ethanol mixtures, and their origin is discussed in detail in section 6. Figure 5.6 suggests that these states are three-frequency states, in which $\omega_1$ is the fast chevron frequency, $\omega_2$ represents the blinking frequency, while the third frequency $\omega_3$ represents the slow modulation frequency. Figure 5.6 shows that these states consist of long intervals consisting of a slowly growing (even) chevron state. Instead of saturating this state becomes unstable to the onset of blinking, which leads to a collapse of the state to small amplitude, followed by a slow regrowth. In the time series shown these collapse events are periodic with period $T_3 = 2\pi/\omega_3$. In section 6 we show that states of this type come about via
Figure 5.6. The repeated transient state in a $^3$He-$^4$He mixture with $S = -0.1$, $\sigma = 0.6$, $\tau = 0.03$ and $\Gamma = 10$ at several values of $R$. The time series $w(x = 0.87\Gamma, z = 0, t)$ suggest the presence of three frequencies, with the lowest frequency ($\omega_3$) increasing from zero as $R$ increases from $R \approx 1971.5$.

Figure 5.7. The time series $w(x = 0.87\Gamma, z = 0, t)$ for a $^3$He-$^4$He mixture with $S = -0.1$, $\sigma = 0.6$, $\tau = 0.03$ and $\Gamma = 10$ at several values of $R$ showing the transition from the repeated transient state in fig. 5.6 to a symmetric periodic blinking state at $R = 1981$. 
a very natural mechanism, and argue that quasiperiodic states with three frequencies should not be thought of as rare even in the absence of any symmetries that in other systems prevent frequency locking.

As $R$ is increased $\omega_3$ gradually increases, but drops out from the time series between $R = 1980.0$ and $R = 1981.0$ (fig. 5.7). We identify this transition with a Hopf bifurcation, and note that fig. 5.7 indicates that this bifurcation is supercritical, i.e., viewed in the direction of decreasing $R$ this bifurcation creates a stable three-frequency state from a stable two-
Figure 5.10. Concentration contours during (a) the low frequency part of the 'fish' state in fig. 5.9, and (b) during the high frequency phase that follows it. The localized state in (a) settles against the left sidewall forming temporarily a confined left-traveling wave shown in (b).

Figure 5.11. (a) The modulation period $T_3 \equiv 2\pi/\omega_3$ of the repeated transients when $S = -0.1$, $\sigma = 0.6$, $\tau = 0.03$, $\Gamma = 10$ in units of $T_0 \equiv 2\pi/\omega_c$ as a function of the Rayleigh number $R$. (b) The corresponding Nusselt number.

frequency state, with no observable hysteresis. Figure 5.8 shows that this two-frequency state is a symmetric blinking state, and traces the evolution of this state towards larger values of $R$. The figure shows that at $R = 1995$ the blinking has become asymmetric and nonperiodic, while the time series for $R = 2000$ and $R = 2005$ may be periodic, but are strongly asymmetric. In contrast, when $R = 2010$ the blinking becomes once again periodic and symmetric. None of these transitions appear to be hysteretic. At yet larger values of $R$ these states again evolve into the fish state, followed by a transition to stationary convection. Figure 5.9 shows an example of the fish state just prior to this transition. An examination of the spatial structure of the waves shows that there is an instant during which the waves takes the form of a small amplitude chevron state filling the container. This state is unstable, however, with the waves at one end growing at the expense of those at the other. At this point the system shifts into a new state, one in which the waves are spatially confined towards one side, with no waves at the other (see fig. 5.10a). This transition is marked by a dramatic drop in the frequency $\omega_1$. This change in frequency in turn increases the Nusselt number, and does so despite the fact that the waves no longer fill the whole domain. As time proceeds this confined state settles next to the boundary (see fig. 5.10b) and during this process both $w(x = 0.87\Gamma, z = 0, t)$ and $\omega_1$
increase rapidly, although the Nusselt number falls, developing a prominent shoulder. The state is now so confined that waves start to regrow at the other sidewall, restoring the small amplitude extended chevron state. As this occurs the amplitude of the confined state falls and the state begins to expand, the frequency $\omega_1$ increasing towards the Hopf frequency $\omega_c$. The appearance of a dynamically confined state implies that the time series shown in fig. 5.9 cannot be understood in terms of the type of theory summarized in section 2. Indeed the value of $\epsilon$ corresponding to fig. 5.9, $\epsilon = 2.35 \times 10^{-2}$, indicates that for these values of $R$ the dynamics of the system are no longer necessarily dominated by the sidewalls (since $\Gamma \gtrsim \epsilon^{-1/2}$). Indeed our calculations are consistent with the suggestion that the fish states first become possible when $\Gamma \sim \epsilon^{-1/2}$, a scaling suggested by Ginzburg-Landau theory.

Figure 5.11 reveals a fundamental property of the repeated transient state: when $R$ is decreased from $R = 1974$ the period $T_3$ increases rapidly and apparently diverges at $R \approx 1971.5$; no stable solutions are present for smaller values of $R$. Thus the three-frequency repeated transient state represents the first nontrivial state of this system.

The final case we have considered is $S = -0.5$. Here $R_c = 2643.43$, $\omega_c = 12.836$. The primary instability is again to an even chevron, but this time we find strongly irregular dynamics already quite close to onset (fig. 5.12). The time series in fig. 5.12 is best described as intermittent repeated transients, in which the final collapse event may be preceded by several spatially symmetric bounces before the onset of the symmetry-breaking instability that disrupts the state and leads to the temporary formation of a growing confined state at one of the sidewalls, much as already described for $S = -0.1$. As this state grows it shrinks in lateral extent, until it triggers another collapse event that permits waves to regrow at the other sidewall. The decaying symmetric blinking state that results reestab-
lishes a small amplitude chevron state that then regrows on a much longer timescale. The spatially symmetric bounces are associated with relatively sharp peaks in the Nusselt number, while the symmetry-breaking collapse events produce bursts in the Nusselt number that are markedly asymmetric, much as in the fish state discussed above. It should be noted that the burst state coexists with a stable odd parity chevron state with a superposed small amplitude temporal modulation that is exactly out of phase in the two halves of the domain. This modulation has a complex waveform but appears to be periodic in time. This state is therefore a symmetric blinking state, and indeed with decreasing $R$ one finds that the modulation disappears and (at $R = 2606$) a stable odd parity chevron is recovered. This multistability greatly complicates the behavior of the system, and appears to be a consequence of increasing subcriticality of the primary bifurcations with decreasing separation ratio.

The frequency of the burst-like events in the Nusselt number increases with $R$ (fig. 5.13). A periodic sequence of bursts, produced by a symmetric albeit complex state, is shown in fig. 5.14. Perhaps the most remarkable time series of all is shown in fig. 5.15 for $R = 2750$. This series apparently shows an irregular switching between two states, a large amplitude state with a relatively low $\omega_1$, and a small amplitude state with a large $\omega_1$. The former is a wave confined spatially to one or other sidewall, while the latter is a more-or-less symmetric extended chevron-like state. This pulsating state loses stability to a symmetry-breaking instability which kills off the waves at one of the sidewalls, forming a spatially confined state that slowly drifts to the opposite sidewall (cf. fig. 5.10). Once in contact with the wall the increased dissipation reduces its amplitude and permits waves to regrow at the other wall. The growing blinking state appears to glue with its mirror image, briefly forming a symmetric pulsating chevron, which is unstable in turn to the original symmetry-breaking instability. These gluing transitions can be seen in figs. 5.7 and 5.8 as well.

**Figure 5.13.** As for fig. 5.12 but showing the time series corresponding to $R = 2655$. 

```latex
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_13}
\caption{As for fig. 5.12 but showing the time series corresponding to $R = 2655$.}
\end{figure}
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3. Oscillatory Binary Fluid Convection

In fig. 5.16 we show a strongly confined traveling wave found at $R = 2900$. A wave of this type evolves from the fish state when the localized low frequency state comes to rest against a sidewall, but does not collapse. The resulting wave should be interpreted in terms of a stationary front separating an exponentially small amplitude wave throughout most of the domain from a finite amplitude wavetrain next to the left sidewall. Consequently fig. 5.16 represents a dynamically localized wave, instead of the kinematic localization described by steady states of equations (2.3) with $|v| \gg |w| > 0$. In the latter the amplitude of the counterpropagating is never zero, and a small amplitude right-traveling wave is always visible near the right sidewall; the resulting state is nothing but a strongly asymmetric chevron. In contrast fig. 5.16 shows no evidence of a right-traveling wave in any part of the container except perhaps right next to the left sidewall. Such waves are described well by a single complex Ginzburg-Landau equation with a drift (see eqs. (2.4) with $B \equiv 0$), and hence become possible once $|\epsilon| \gtrsim \Gamma^{-2}$. Theory based on this equation predicts

Cross, 1986, 1988; Tobias, Proctor, and
Knobloch, 1998] that with increasing $R$ the front gradually moves towards the right, but in the present case the strong nonlinear dispersion forces the frequency towards zero; once this occurs the resulting non-oscillatory state begins to expand towards the right by adding (steady) rolls and thereby expelling the lateral concentration gradient set up by the confined traveling wave. The process of adding rolls terminates once this lateral gradient is sufficiently strong, and results in the formation of a spatially confined but steady state (see fig. 5.17). Such confined steady states have also been found for other values of $R$ (fig. 5.18) indicating that the confined states created by this process are in general non-unique. All of these states are numerically stable. However, to our knowledge no comparable states have been observed in any experiment.

Figure 5.19 shows the chaotic blinking state at $R = 2665$ that coexists stably with the bursts shown in fig. 5.14. As already mentioned this state is based on an odd parity chevron, in contrast to fig. 5.14 which is based on an even chevron.
3. Oscillatory Binary Fluid Convection

Figure 5.17. A stable spatially confined steady state at $R = 3050$ when $S = -0.5$, $\sigma = 0.6$, $\tau = 0.03$, $\Gamma = 10$.

Figure 5.18. Other stable spatially confined steady states when $S = -0.5$, $\sigma = 0.6$, $\tau = 0.03$, $\Gamma = 10$. The corresponding value of $R$ is indicated at the right of each panel.

Figure 5.19. A stable chaotic blinking state at $R = 2665$ based on an odd parity chevron that coexists with the bursts shown in fig. 5.14.
5.2 Water-ethanol mixtures in $16 \leq \Gamma \leq 17.25$ containers

In this section we describe our results [Batiste, Knobloch, Mercader, and Net, 2001a; Batiste, Net, Mercader, and Knobloch, 2001b] for the parameter values used by Kolodner [1993] in his experiments, $S = -0.021$, $\sigma = 6.97$, $\tau = 0.0077$, focusing on Rayleigh numbers near threshold (i.e., $|\epsilon| \ll 1$) and on their dependence on the aspect ratio $\Gamma$.

Figure 5.20 summarizes the evolution for different values of $\Gamma$ of the midplane vertical velocity $w(x = 0.87\Gamma, z = 0, t)$ obtained by integration over $2000t_d$ after an initial transient has (almost) died out. The figure illustrates the sensitive dependence on the aspect ratio for comparison with figure 2 of Kolodner [1993], as well as the long integration times required to get reliable results. The high frequency uniform amplitude states correspond to nonlinear time-periodic chevron states such as the one shown in fig. 5.21. The figure shows that although the temperature departure $\theta$ from the conduction profile remains sinusoidal in space at this value of $\epsilon$ this is not so for the concentration departure $C$. As explained by Barten, Lücke, Hort, and Kamps [1989] this is a consequence of the small value of $\tau$. Note in particular that regions of high and low concentration departure are separated by open contours, in contrast to the temperature field. The temporary straightening of these meandering concentration contours in the cell center every half period accompanies the splitting of the central concentration roll into two. Both these properties of the concentration field are absent from the Ginzburg-Landau description of this system.

In order to understand the origin and character of the nonperiodic states seen in fig. 5.20 we show in figs. 5.22 and 5.23 detailed results for $\Gamma = 16.25$ and $\Gamma = 16$, and $\epsilon$ in the vicinity of $\epsilon = 0$. These plots show that the initial bifurcation to the chevron state is subcritical, in agreement with the prediction for standing waves in a horizontally unbounded layer [Clune and Knobloch, 1992; Schöpf and Zimmermann, 1989]. Figure 5.22 for $\Gamma = 16.25$ shows that the (even) chevron state can equilibrate at finite amplitude ($R = 1775.5$). However, with increasing $\epsilon$ the stable chevrons lose stability in a supercritical Hopf bifurcation. This bifurcation introduces a new frequency into the system, seen in fig. 5.22 ($R = 1776.0$) as an oscillation in the amplitude of $w(x = 0.87\Gamma, z = 0, t)$; the resulting state is a symmetric periodic blinking state. Such states may set in already for $\epsilon < 0$; as $\epsilon$ increases the blinking amplitude increases and the blinking becomes nonperiodic.

The results for $\Gamma = 16$ (fig. 5.23) are quite different. In this case no subcritical stable chevrons are observed, and instead the first nontrivial state of the system appears to be a three-frequency state ($R = 1777.2$). We shall see in section 6 that such states are entirely natural in systems of this type. Figure 5.23 also shows that with increasing $\epsilon$ this state evolves into one increasingly like Kolodner’s repeated transients, with a growth
3. Oscillatory Binary Fluid Convection

Figure 5.20. Water-ethanol mixtures with $S = -0.021, \sigma = 6.97, \tau = 0.0077$: an overview of the aspect ratio dependence of the equilibrated states near onset, in terms of the vertical velocity $w(x = 0.87\Gamma, z = 0, t)$ for comparison with figure 2 of Ref. [8]. The numbers at the right give the values of $10^4\epsilon$ and correspond to the solid dots in fig. 4.7.

Figure 5.21. Periodic even-parity chevron for $\Gamma = 16.25$ and (a,c) $R \equiv R_e = 1776.3$, (b,d) $R = 1775.5 (\epsilon = -4.5 \times 10^{-4})$ in terms of the contours of (a,b) the temperature perturbation $\theta(x, z, t)$ and (c,d) the concentration perturbation $C(x, z, t)$ of the denser component, with time increasing upwards in intervals of $0.2t_d$. The temperature contours at $R = 1775.5$ resemble the eigenfunction but the concentration contours do not.
Figure 5.22. Time series $w(x = 0.87\Gamma, z = 0, t)$ for $\Gamma = 16.25$ and different values of the Rayleigh number $R$. Stable chevrons are present for $R = 1775.5$ ($\epsilon = -4.5 \times 10^{-4}$), but give way to symmetric periodic blinking states when $R = 1776$ ($\epsilon = -1.7 \times 10^{-4}$) with no detectable hysteresis. At larger $R$ (e.g., $R = 1778$) the blinking states are asymmetric.

Figure 5.23. Time series $w(x = 0.87\Gamma, z = 0, t)$ for $\Gamma = 16$ and different values of the Rayleigh number $R$. The first finite amplitude state is a three-frequency state at $R = 1777.2$ ($\epsilon = -1.8 \times 10^{-4}$). This state gives way gradually and without detectable hysteresis to repeated transient states near $\epsilon = 0$ and then to symmetric periodic blinking states in a hysteretic transition between $R = 1778.5$ ($\epsilon = 5.5 \times 10^{-4}$) and $R = 1779$ ($\epsilon = 8.3 \times 10^{-4}$). The state at $R = 1782$ ($\epsilon = 2.5 \times 10^{-3}$) appears to have period-two modulation.
phase that becomes progressively shorter, before a (hysteretic) transition to the symmetric periodic blinking state takes place ($1778.5 < R < 1779$). With further increase in $\epsilon$ this state gradually evolves into a chaotically blinking state much as in fig. 5.22, as anticipated in section 2 and observed in experiments [Kolodner, Surko, and Williams, 1989; Steinberg, Fineberg, Moses, and Rehberg, 1989].

Stable subcritical chevrons were found for all other values of $\Gamma$ explored except $\Gamma = 16$ and $\Gamma = 17$, but their range of stability is usually narrow and may lie in $\epsilon < 0$. This may explain why stable chevrons have only rarely been found in experiments [Steinberg, Fineberg, Moses, and Rehberg, 1989]. In any case these lose stability almost immediately to a symmetric periodic blinking state, which then becomes asymmetric and/or chaotic (fig. 5.22). Thus the first state encountered on raising $\epsilon$ through $\epsilon = 0$ is typically a complex state even when $\Gamma \neq 16, 17$. However, symmetric periodic blinking states are present at onset near the mode crossing point at $\Gamma \approx 16.8$ (see figs. 5.20, 5.24), in accord with theoretical expectation: the onset of blinking is a consequence of the interaction between odd and even chevrons in the nonlinear regime. The blinking period we find, $\approx 90t_d$, is comparable to the period measured in the experiments when $\Gamma = 17.63$, viz. 8000s, since $t_d = 84.3s$ [Kolodner, 1993] and the $\Gamma = 17.63$ case behaves much like $\Gamma = 16.63$ (section 2).
Figure 5.24 explores the evolution of the blinking states at larger values of $\epsilon$ when $\Gamma = 16.8$ using time series for the vertical velocity at mirror points in the two halves of the container. The solutions are in general asymmetric with respect to the middle of the container, and may be periodic (as for $R = 1783$, for example) or chaotic (as for $R = 1784$). The figure also shows that with increasing $\epsilon$ the left (right) traveling waves become increasingly confined to the left (right) half of the container, leading to characteristic time series of the type shown for $R = 1787$ and $R = 1788$. Note that the former is strongly spatially asymmetric, while the latter is symmetric, with both states being periodic. Thus transitions that break and restore the symmetry in the vertical midplane may occur repeatedly, with the symmetric states at $R = 1786$ and $R = 1788$ separated by an asymmetric one, suggesting the presence of a cascade of gluing bifurcations, cf. Arnéodo, Coullet, and Tresser [1981]. Note that the time series for $R = 1789$ ($\epsilon = 0.006$) is very similar to the fish state observed by Kolodner for $\Gamma = 16.75$ and $\epsilon = 0.0111$, even to the extent of capturing the strong amplitude dependence of the chevron frequency within this state (cf. section 5.1). Moreover, the computed period of this state, $305t_d$ (see fig. 5.24) corresponds almost exactly to that measured by Kolodner in the experiment (see fig. 1.1(f)).

Figures 5.25 and 5.26 show the corresponding results for $\Gamma = 17.25$ and $\Gamma = 17$, for comparison with figs. 5.22 and 5.23, respectively. In both cases the results are very similar. Thus when $\Gamma = 17.25$ the first stable state is an (odd) chevron, which undergoes a supercritical bifurcation to a blinking state when $\epsilon$ is still negative. With increasing $\epsilon$ the time series gradually become more complex, and begin to resemble the states we have called repeated transients (see fig. 5.25). However, these states develop from small amplitude blinking states with increasing $R$ and hence can be unambiguously distinguished from them. In fact it is likely that the states shown in fig. 5.25 (at $R = 1775.5$, for example) are the result of another Hopf bifurcation from the blinking state, but this time with $R$ increasing, not decreasing. Figure 5.25 shows that as $R$ increases further these states gradually become chaotic, although at $R = 1779$ the time series is again periodic. The results for $\Gamma = 17$ likewise resemble those for $\Gamma = 16$. In particular the first nontrivial state is again a repeated transient, this time consisting of a slowly but exponentially growing odd parity chevron, followed by the characteristic oscillatory collapse. In fact the results for $\Gamma = 17$ provide a somewhat clearer illustration of the origin of the three-frequency state, since they suggest that the oscillatory collapse phase connects a larger amplitude chevron state with a smaller amplitude chevron which then regrows again into the larger amplitude state. The transition from this state to the blinking state appears to be again hysteretic, while the largest Rayleigh number solution ($\tilde{R} = 1780$) appears to be periodic but with a period that is three times the basic blinking period. The only substantive difference between the results of figs. 5.23 and 5.26 is that the parity of the chevron
Figure 5.25. Time series $w(x = 0.87\Gamma, z = 0, t)$ for $\Gamma = 17.25$ and different values of the Rayleigh number $R$. Stable chevrons are present for $R = 1774.3$ ($\epsilon = -5.2 \times 10^{-4}$), but give way to symmetric periodic blinking states when $R = 1774.5$ ($\epsilon = -4.1 \times 10^{-4}$) with no detectable hysteresis. At larger $R$ (e.g., $R = 1776$) the blinking state evolves to a three-frequency state superficially resembling a repeated transient.

Figure 5.26. Time series $w(x = 0.87\Gamma, z = 0, t)$ for $\Gamma = 17$ and different values of the Rayleigh number $R$. The repeated transient state gives way to a periodic blinking state in a hysteretic transition between $R = 1777$ ($\epsilon = 3.7 \times 10^{-4}$) and $R = 1777.5$ ($\epsilon = 6.5 \times 10^{-4}$). The state at $R = 1776.1$ ($\epsilon = -1.4 \times 10^{-4}$) eventually decays, while that at $R = 1780$ ($\epsilon = 2.1 \times 10^{-3}$) appears to have period-three modulation. The results follow closely the sequence shown in fig. 5.23.
state involved is different. These results confirm Kolodner’s experimental results and the theoretical prediction [Dangelmayr, Knobloch, and Wegelin, 1991] that the spatio-temporal dynamics of this system should be periodic with respect to $\Gamma$ with a period of $\pi/k_c$, where $k_c \approx \pi$ is the wavenumber obtained from linear theory (cf. section 2). Thus increasing the aspect ratio by one allows the system to insert an extra roll thereby changing the parity of the basic state. As in the $^3$He-$^4$He mixtures the wavelength of the rolls remains remarkably uniform across the container despite the substantial changes in amplitude that occur as a result of the dynamics of these states.

The explanation of these results is the subject of the next section.

6 Origin of the Repeated Transients

In this section we investigate the model equations

\[ \dot{v} = (-\nu + cz^2)v - \delta|v|^2v, \]
\[ \dot{z} = (\mu + az^2 - z^4)z - |v|^2z, \]

(6.1)

constructed to retain the main properties of the partial differential equations. Here $z$ refers to the amplitude of the chevron state (either even or odd) and is taken to be a real quantity despite the fact that the chevron states are in fact time-periodic. We justify this approximation using fig. 5.20 which shows that, for the parameter values considered, the chevron frequency $\omega_1$ is high compared to the blinking frequency $\omega_2$ or the slow frequency $\omega_3$ associated with the repeated transients. This assumption has the considerable advantage in that it removes one frequency from the system, and can be considered to be the result of averaging over the fast chevron frequency. Consequently pure chevron states correspond to the solutions of the equation

\[ \dot{z} = (\mu + az^2 - z^4)z, \]

(6.2)

and we take $\mu \propto R - R_c(\Gamma)$ to be a real parameter, with the coefficient $a$ also real. In view of the results of section 5.2 we take $a > 0$ so that the primary bifurcation to chevrons is subcritical, with a saddle-node bifurcation (hereafter SN) occurring at $z^2 = a/2$. The stability of these states with respect to perturbations in the form of chevrons of the same parity is therefore given by the linearization of equation (6.2) about the solution $z = z_0$ satisfying $\mu + az_0^2 - z_0^4 = 0$. We denote this eigenvalue by $\lambda$. It follows that when $a < 0$ this eigenvalue is always negative (stable) while if $a > 0$ it is positive (unstable) on the subcritical branch and becomes negative above the saddle-node bifurcation.

The variable $v$ represents perturbations transverse to the chevron invariant subspace, and is complex because these perturbations are destabilized at a secondary Hopf bifurcation, hereafter $H_2$. As a result the coefficients $\nu,$
$c$ and $\delta$ are all complex. This Hopf bifurcation is responsible for the onset of blinking. In the model the amplitude of the blinking is given by

$$\dot{y} = (-\nu_R + c_R z^2)y - \delta_R y^3$$

(6.3)

and its frequency by the decoupled equation

$$\dot{\theta} = -\nu_I + c_I z^2 - \delta_I y^2.$$  

(6.4)

Here $v \equiv ye^{i\theta}$ and the subscripts $R$ and $I$ denote real and imaginary parts, respectively. In these equations the important parameter is $\nu_R \equiv \nu_R(\Gamma) > 0$ and we take $c_R > 0$. Because of the decoupling of $\theta$ from the equations for $y$ and $z$ the resulting model is simple to analyze. Within the model the symmetry $y \to -y$ represents evolution in time by half the blinking period so that solutions with opposite signs of $y$ are in fact identical modulo time translation. The pure chevrons $(y, z) = (y_0, z_0)$ begin to blink when $z_0^2 = \nu_R/c_R$ and do so with frequency $-\nu_I + c_I z_0^2$, the resulting blinking states take the form $(y, z) = (y_0, z_0)$, provided $y_0^2 > 0, z_0^2 > 0$. The stability of these states is described by a quadratic dispersion relation. This relation shows that the blinking states either set in (supercritically) from the larger amplitude chevron branch (hereafter A), or from the smaller amplitude branch (hereafter B). In the former case the chevrons acquire stability at the saddle-node bifurcation before losing it again at larger amplitude to stable blinking states. In the $(y, z)$ variables this bifurcation looks like a pitchfork bifurcation, although it is of course a Hopf bifurcation. In the latter case the blinking states are initially unstable but acquire stability at a tertiary Hopf bifurcation $H_3$. This Hopf bifurcation is of vital importance in what follows since it introduces a third frequency, $\omega_3$, into the dynamics of the partial differential equations. Its presence is a direct consequence of the passage of the Hopf bifurcation $H_2$ through the saddle-node bifurcation $SN$ on the chevron branch when $a > 0, c_R > 0$, as originally noted by Guckenheimer [1981]. For a related analysis, also arising in the binary fluid context, see Knobloch and Moore [1990]. In the following we present the corresponding results for the full model equations (6.1). These are summarized in fig. 6.1 for the case in which the three-frequency state created from the blinking state branch is stable. This is always the case when $c_R = 1, \delta_R = 0$ and $a > 0$, and hence for sufficiently small positive values of $\delta_R$ as well. The figure shows the loci of the primary ($H_1$), secondary ($H_2$) and tertiary ($H_3$) Hopf bifurcations, as well as the locus of the saddle-node bifurcations (SN) on the chevron branch. It should be remembered that in the $(y, z)$ variables only the bifurcation $H_3$ remains a Hopf bifurcation, with $H_1$ and $H_2$ represented by pitchfork bifurcations. In addition the figure shows the curve $\gamma$ of global bifurcations at which the limit cycle (corresponding to the three-frequency states) created at $H_3$ disappears by simultaneous collision with the pure chevron states A and B. The location of this line must be determined numerically. An asymptotic calculation of this curve
near the codimension-two point yields the heavy broken line; this line is tangent to $\gamma$ at the codimension-two point, as it must.

![Figure 6.1](image)

**Figure 6.1.** Codimension-one bifurcation surfaces in the $(\mu, \nu_R)$ plane for equations (6.1) with $a = 2.0$, $c_R = 1.0$ and $\delta R = 0.1$. $H_1$: primary (Hopf) bifurcation to the chevron state $(v, z) = (0, z_0)$, SN: saddle-node bifurcation on the chevron state, $H_2$: (secondary) Hopf bifurcation to blinking states $(v, z) = (v_0, z_0)$, $H_3$: (tertiary) Hopf bifurcation from $(v_0, z_0)$ responsible for the appearance of the three-frequency states, and $\gamma$: global bifurcation at which these states disappear. The heavy broken line represents the asymptote to $\gamma$.

Figure 6.2 shows the bifurcation diagrams obtained by traversing the $(\mu, \nu_R)$ plane in fig. 6.1 along the lines $\nu_R = 1.6$ and $\nu_R = 0.7$. These capture the two fundamentally different bifurcation diagrams characterizing the binary mixture. Figure 6.2a shows a small interval of subcritical but stable chevrons, followed by a supercritical pitchfork bifurcation to a state with $y_0 \neq 0$ that represents a blinking state in the physical variables. In the example shown this bifurcation occurs at $\mu < 0$ so that the first stable state just above onset $(\mu = 0)$ is a finite amplitude blinking state. In contrast, in the case shown in fig. 6.2b the first stable state encountered beyond $\mu = 0$ is a finite amplitude periodic state which we identify with the three-frequency repeated transient state discovered by Kolodner. Figure 6.2c shows the time series corresponding to this state when $\mu = -0.21$. These oscillations represent the low frequency component of the three-frequency state, i.e., the repeated transient state with the frequencies $\omega_1$ and $\omega_2$ filtered out. Observe that during the growth phase of the variable $z$ the variable $y$ vanishes, indicating that the growing state is a pure chevron; $y$ becomes nonzero only during the collapse phase, indicating that the collapse is triggered by a symmetry-breaking instability (i.e., the loss of stability of the growing chevron). Figure 6.3 shows similar behavior obtained from the partial differential equations when $\Gamma = 16$: during the growth phase $w(0.13\Gamma, 0, t) = w(0.87\Gamma, 0, t)$, indicating a growing
3. Oscillatory Binary Fluid Convection

Figure 6.2. (a,b) The bifurcation diagrams along the lines \( \nu_R = 1.6 \) and \( \nu_R = 0.7 \) in fig. 6.1. Scenario (a) corresponds to that observed in fig. 5.22 for \( \Gamma = 16.25 \), while (b) corresponds to that observed in fig. 5.23 for \( \Gamma = 16 \). In (b) the open circles indicate the global bifurcation with which the oscillations terminate as \( \mu \) decreases, with the states A and B labeled as in the text. Solid (dashed) lines indicate stable (unstable) solutions. (c) The time series \( y(t) \) (dashed) and \( z(t) \) (solid) for a stable repeated transient when \( \mu = -0.21, \nu_R = 0.7, a = 2.0, c_R = 1.0 \) and \( \delta_R = 0.1 \).

Figure 6.3. Plots of \( w(x = 0.13 \Gamma, z = 0, t) \) vs \( w(x = 0.87 \Gamma, z = 0, t) \) for \( \Gamma = 16, R = 1778.5 \) during (a) the growth and the collapse phases, and (b) during the start of the collapse phase, together with the time series (c) \( w(x = 0.87 \Gamma, z = 0, t) \) and (d) \( w(x = 0.13 \Gamma, z = 0, t) \). Growing symmetric chevrons evolve along the 45° line in (a) but evolve along the orthogonal direction during the collapse phase (cf. figure 8 of Kolodner [1993]). The growth, transition and collapse phases used to construct figs. (a,b) are indicated by vertical dashed lines in figs. (c,d).
chevron, a fact confirmed in fig. 6.3a by the evolution of the system away from the origin along the 45° line. The collapse phase is initiated when the difference between \( w(0.13\Gamma, 0, t) \) and \( w(0.87\Gamma, 0, t) \) begins to grow and the system begins to evolve in a direction orthogonal to the 45° line, much as shown in figure 8a of Kolodner [1993]. With the beginning of the collapse phase one starts to notice the onset of blinking as evidenced in the 180° phase difference between the decaying oscillations in \( w(0.87\Gamma, 0, t) \) and \( w(0.13\Gamma, 0, t) \) (fig. 6.3c,d). The amplitude and the period \( 2\pi/\omega_3 \) of the limit cycle in fig. 6.2c decreases with increasing \( \mu \), with the oscillations disappearing at \( H_3 \). As already mentioned we interpret this transition as the transition from the repeated transient state to the (symmetric) periodic blinking state with increasing Rayleigh number (cf. fig. 4 of Kolodner [1993], where \( H_3 \) is located at \( \epsilon \approx 2 \times 10^{-3} \), i.e., the minimum of the measured “modulation” period). For the model parameters this transition is supercritical, indicating absence of hysteresis. As \( \mu \) approaches closer to the global bifurcation at \( \mu = \mu^* < 0 \), indicated by open circles in fig. 6.2b, the time series remains similar to that shown in fig. 6.2c but the oscillation period \( 2\pi/\omega_3 \) becomes longer, diverging as \( -\ln|\mu - \mu^*| \) for \( \mu \sim \mu^* \), cf. figure 4 of Kolodner [1993]. In fig. 6.4 we show another case, in which the global bifurcation at \( \mu^* \) occurs very close to \( \mu = 0 \). As a result the chevron state grows from almost zero amplitude, and so resembles more closely the repeated transient state discovered by Kolodner. In this case there is almost no hysteresis between this state and the conduction state, and the system behaves as if the primary instability at \( \mu = 0 \) were directly responsible for generating repeated transients. Within the model the corresponding state has all the properties of this state observed in the experiments, except for the (apparent) absence of oscillations during the collapse phase. In fact, if the frequencies \( \omega_1 \) and \( \dot{\theta} \equiv \omega_2 \) determined by equation (6.4) are incorporated, and the quantity \( [z + v_R(t)] \sin \omega_1 t \equiv [z + y(t) \cos \omega_2 t] \sin \omega_1 t \) is plotted instead of \( z \) or \( |v| \), these oscillations are present (fig. 6.5b), and their amplitude depends on the chevron amplitude \( z \) in the manner observed in the experiments. In fact, the time series shown in fig. 6.5b bears a number of qualitative features, including the pointed overshoot at maximum as the mode \( v \) begins to grow and the “ringing down” due to the fact that the variable \( z \) decays more rapidly than \( v \), that are documented in figure 6a of Kolodner [1993]. This time series is not periodic because in general the two (nonlinear) frequencies \( \omega_2 \) and \( \omega_3 \) are incommensurate.

Several remarks are in order.

1. The coefficient \( \delta \) can be zero without qualitative effect on the above scenarios. However, we have chosen \( \delta_R > 0 \) to assure that the solutions remain bounded for all time, and to move the secondary bifurcations away from the saddle-node on the primary chevron branch.

2. The invariance of the plane \( z = 0 \) in the two-dimensional model prevents the formation of a connection between the large amplitude
3. Oscillatory Binary Fluid Convection

Figure 6.4. (a) As for fig. 6.2b but with $\nu_R = 0.15$, $a = 2.0$, $c_R = 1.0$ and $\delta_R = 0.2$, showing a global bifurcation very close to $\mu = 0$. (b) The corresponding time series when $\mu = 0.02$.

Figure 6.5. (a) As for fig. 6.4b but over a longer time interval. (b) The time series for $[z + v_R(t)] \sin \omega_1 t$ when $\nu_I = 0.8$, $c_I = 0$, $\delta_I = 0$ and $\omega_1 = 20$. Note the exponential growth during the chevron phase, followed by an overshoot when the blinking instability sets in, and the ringing down during the subsequent collapse phase. The time series resembles closely that in fig. 6a of Kolodner [1993]. (c) $|v(t)|$ for chaotic repeated transients from equations (6.5) with $\epsilon_1 = 0.1 + 0.1i$, $f_I = z^2$. 
The model (6.1) described above is completely consistent with the two scenarios for generating blinking states identified in the simulations of both $^3$He–HeliumIV and water–ethanol mixtures. For example, in the scenario depicted in fig. 5.22 the blinking sets in via a supercritical Hopf bifurcation above the saddle-node bifurcation, and does so already for $\epsilon < 0$. Consequently there is only a narrow range of $\epsilon$ between this bifurcation and the saddle-node bifurcation with stable chevrons, before blinking sets in. The blinking frequency $\omega_2$ is quite small because the chevron amplitude at which the Hopf bifurcation takes place is small [Dangelmayr and Knobloch, 1987, 1991]. In contrast, the results for $\Gamma = 16$ (fig. 5.23) and $\Gamma = 17$ (fig. 5.26) are entirely consistent with the second scenario, i.e., that the secondary Hopf bifurcation to blinking states now occurs below the saddle-node bifurcation, thereby eliminating the stable chevrons entirely. Moreover, our results for, say, $R = 1777.2$ (fig. 5.23) and $R = 1776.5$ (fig. 5.26) are suggestive of a quasiperiodic state with three independent frequencies such as might be expected from the tertiary Hopf bifurcation $H_3$ on the branch of blinking states identified in the model. Indeed, our
calculations are consistent with the conjecture that the bifurcations SN and H₂ on the chevron branch coincide at an aspect ratio \( \Gamma \) somewhere between 16 and 16.25. Note that the observed period associated with the third frequency is about 1000\( t_d \). Such low frequencies are characteristic of the scenario proposed in fig. 6.1. Moreover, this scenario predicts that the corresponding modulation period should increase rapidly with decreasing \( \epsilon \), i.e., as \( \epsilon \downarrow \epsilon^* \), in accord with the experimental observations (see figure 4 of Kolodner [1993]). Figure 5.23 also suggests that the repeated transients observed by Kolodner evolve from this three-frequency state as \( \epsilon \) increases from \( \epsilon^* \), a suggestion that is confirmed in fig. 5.26 where the three-frequency states look like Kolodner’s repeated transients from the very beginning. In both cases periodic blinking states are observed only after a (hysteretic) transition from the three-frequency repeated transients. Consequently the branch of blinking states only acquires stability at H₃ and these states therefore blink with finite amplitude when they first appear, resulting in a longer blinking period than at H₂, typically 100\( t_d \) (compare fig. 5.23 at \( R = 1779 \) with fig. 5.22 at \( R = 1776 \)). This period is also comparable to the period observed in the experiments. With further increase in \( \epsilon \) the blinking state appears to undergo period-doubling as suggested by the time series for \( R = 1782 \) in fig. 5.23 (cf. Hirschberg and Knobloch [1996]; Knobloch [1996]), and gradually becomes more and more chaotic. Indeed, the time series for \( R = 1780 \) in fig. 5.26 suggests a period-three blinking state.

Available theory predicts (section 2) that the blinking states terminate in another global bifurcation at which a hysteretic transition to a single frequency localized state takes place. This state consists of waves that travel under a stationary envelope attached to one or other lateral wall. It is likely that the period-doubling transitions etc. are associated with this global bifurcation. Finally, the fact that we have found the repeated transients only in the vicinity of \( \Gamma = 16 \) and \( \Gamma = 17 \), i.e., for aspect ratios differing by \( \approx 1 \), is also consistent with theoretical expectation (section 2), and indeed the experiments as well [Kolodner, 1993].

The presence of the global bifurcation in which the repeated transients first appear, established in figs. 6.2b and 6.4a, suggests that under appropriate conditions the repeated transients may be chaotic. As already noted the frequency \( \omega_3 \) decreases to zero as \( \epsilon \downarrow \epsilon^* \). As this occurs the three-frequency states approach simultaneously the unstable large and small amplitude chevron states A and B. The character of the repeated transient when \( \epsilon \approx \epsilon^* \) is determined by the leading eigenvalues of A and B in the chevron fixed point subspace, hereafter \(-\lambda_A < 0 \) and \( \lambda_B > 0 \), and the leading eigenvalues in the perpendicular direction. If the latter are real, \( \alpha_A > 0 \) and \(-\alpha_B < 0 \), say, and \( \rho \equiv \alpha_B \lambda_A / \alpha_A \lambda_B > 1 \), the repeated transients will remain periodic and stable all the way to \( \epsilon^* \), where the period diverges and the global bifurcation takes place [Batiste, Knobloch, Mercader, and Net, 2001a]. In contrast, when \( 0 < \rho < 1 \), the periodic oscillations necessarily lose stability before the global bifurcation at \( \epsilon^* \). Similar results obtain in the
case where the leading stable symmetry-breaking eigenvalue at B is complex, viz. \(-\alpha_B + i\omega_B, \alpha_B > 0\), as suggested by the simulations. In this case stable periodic oscillations will persist down to \(\epsilon^*\) if \(\rho > 1\), but if \(0 < \rho < 1\) complex dynamics of Shil’nikov type will be present. Possible chaotic repeated transients resulting from this mechanism are shown in fig. 6.6. In fact figs. 5.23 and 5.26 suggest that the leading unstable eigenvalues \(\alpha_A\) and \(\lambda_B\) are also complex; this is to be expected since the bifurcations at \(H_1\) and \(H_2\) are in fact both Hopf bifurcations. In the following we do not consider the resulting complications further.

![Figure 6.6. A chaotic repeated transient in a \(^3\)He-\(^4\)He mixture with stress-free and fixed temperature boundary conditions at \(x = 0, \Gamma\) and \(R = 2025, S = -0.1, \sigma = 0.6, \tau = 0.03, \Gamma = 10\).](image)

When \(\lambda_B\) is real a trajectory escaping from B describes an exponentially growing chevron state. This growth phase, including the states A and B, is clearly visible in the time series for \(R = 1776.2\) (fig. 5.26). When the growing chevron reaches the vicinity of A it becomes unstable to symmetry-breaking oscillations which take it back near B. This is the collapse phase of the repeated transient state (compare figs. 6.2c and 6.4b with fig. 6.3). The frequency of the decaying oscillations observed in the time series in figs. 6.3c,d is given by \(\omega_B\). This frequency will in general be of the same order as the blinking frequency associated with the branch of blinking states when these bifurcate from the small amplitude chevron B, but quite different from (and in general larger than) the blinking frequency of the stable blinking states beyond \(H_3\), cf. Kolodner [1993]. This observation explains the coincidence of the period of the blinking states and of the oscillations during the collapse phase of the repeated transient also noted by Kolodner. Note also that since the repeated transient state visits the states A and B whose amplitude decreases (resp., increases) as \(\epsilon\) becomes more negative the modulation amplitude along the branch of three-frequency states should decrease towards the end of the branch. This is seen quite dramatically in fig. 5.23. Moreover, since \(\alpha_B\) decreases as \(\epsilon\) decreases (it passes through zero at \(H_2\), i.e., at \(\epsilon = \epsilon_2\)) the collapse becomes slower and slower, as also seen in fig. 5.23, but is still finite when the three-frequency states disappear in the global bifurcation at \(\epsilon^*\) (since \(\epsilon_2 < \epsilon^* < 0\)) and the system
makes a hysteretic transition to the conduction state. The fact that $\alpha_B$ decreases with $\epsilon$ makes it likely that the Shil’nikov condition $0 < \rho < 1$ holds at $\epsilon^*$, resulting in chaotic repeated transients prior to their disappearance [Knobloch, 1996]. This possibility apparently does not occur in figs. 5.23 and 5.26 although it does in fig. 6.6 and may occur in the experiments. In any case even longer time series would be required to test this prediction. Note that since $\lambda_B \propto |\epsilon|$ only periodic repeated transients will occur if $\epsilon^* \approx 0$, although even in this case there may be a few bifurcation bubbles with chaotic dynamics, as in Knobloch, Moore, Toomre, and Weiss [1986]. In fact Kolodner [1993] notes that the growth phase of the repeated transient is inversely proportional to $\epsilon$, in accord with the above scenario. It is of interest that the repeated transients are most likely to be chaotic just prior to their extinction, as $\epsilon$ decreases.

We note, finally, that despite the fact that the repeated transients are three-frequency states they are not necessarily structurally unstable. In the model this is because we have averaged out the chevron frequency $\omega_1$. However, even if we had not done so, genuinely three-frequency states can be observed in open parameter regions [Grebogi, Ott, and Yorke, 1983] despite the Ruelle-Takens theorem [Ruelle and Takens, 1971a,b].

7 Discussion

In this article we have described a theoretical approach to understanding the dynamics due to an oscillatory instability in moderately large domains, and confronted the predictions of this theory with the results of direct numerical simulations of oscillatory convection in binary mixtures in two dimensions. For the latter we have used the experimentally relevant boundary conditions and explored in detail the parameter values characteristic of $^3$He-$^4$He and water-ethanol mixtures. We have seen that the competition between odd and even modes in systems of this type takes one of two basic forms, depending on the separation and aspect ratios (see fig. 4.1). When $|S|$ is small (i.e., close to $S_{TB}$) and $\Gamma$ not too large the mode interaction takes the form familiar from Rayleigh-Bénard convection with non-Neumann boundary conditions: the neutral curves $R(\Gamma)$ divide neatly between different families and there is no intermingling among them (see fig. 4.5a). Each family consists of a pair of braided neutral curves, one for an odd mode and the other for an even mode, with each family well separated from the next, at least for the low-lying families. The crossings between odd and even modes within each family are structurally stable because of their different parity. At fixed $\Gamma$ and large enough $|S|$ the situation is quite different (see fig. 4.5c). There are now no distinct families of neutral curves and all modes (including like-parity modes) cross. These mode crossings are all structurally stable, either because the modes have opposite parity, or
because their frequencies at the mode crossing are nonresonant. The transition between these two types of behavior occurs via 1:1 resonant mode interactions as illustrated in fig. 4.3. These interactions allow mode crossings between like-parity modes belonging to different families and hence are responsible for the transition between the neutral stability curves in figs. 4.5a.c. Likewise at fixed $S$ the neutral stability curves are braided when the aspect ratio $\Gamma$ is not too large but with increasing $\Gamma$ nonresonant crossings between like-parity modes appear (cf. fig. 4.3), as anticipated by Hirschberg and Knobloch [1997]. On the basis of our calculations we have made several conjectures about the behavior of the neutral stability curves and corresponding frequencies and eigenfunctions for large aspect ratios. These bear out the picture of large aspect ratio systems put forward by Landsberg and Knobloch [1996] with two significant clarifications. First, we have found that if the separation ratio is small there is a substantial range of aspect ratios within which the first odd and even modes to set in are separated from the next pair by a significant gap in Rayleigh number. In this range the existence of the gap justifies the reduction of the partial differential equations to amplitude equations for the first odd and even modes using a procedure called center-unstable manifold reduction [Armbruster, Guckenheimer, and Holmes, 1989]. We have presented examples of what these modes look like. However, for sufficiently large aspect ratios or sufficiently large separation ratios this gap disappears, and odd and even modes from different families are selected in succession. In this case the reduction to a pair of amplitude equations continues to be valid near all crossing points between odd and even modes, but it is no longer clear whether such a description captures the behavior of the system for all intervening aspect ratios. Second, we have found that the frequency difference between the competing odd and even parity modes scales like $\Gamma^{-2}$ for large $\Gamma$ instead of the expected $\Gamma^{-1}$ behavior. This observation strengthens the argument in favor of the theory put forward by Landsberg and Knobloch [1996] and summarized in section 2.

The results of the direct numerical simulations reveal a complex sequence of transitions among states we have called chevrons, blinking states and repeated transients as the aspect ratio or the applied Rayleigh number varies. We have seen that

- The primary bifurcation to convection is subcritical, in agreement with the prediction for standing waves in a horizontally unbounded layer [Clune and Knobloch, 1992]. As a result for some aspect ratios stable chevrons can be present, but are most likely observed for negative values of $\epsilon$.

- Stable small amplitude blinking states set in when stable chevrons lose stability at a secondary Hopf bifurcation.

- For many (though not most) aspect ratios the first nontrivial state of
the system is a three-frequency repeated transient state. The transition to this state is hysteretic since $\epsilon^* < 0$ but for many parameter values $\epsilon^*$ is so close to zero as to make the detection of hysteresis highly unlikely. Indeed, in some experiments no hysteresis was found [Kolodner and Surko, 1988]. The three-frequency states appear via a global bifurcation and acquire the characteristic behavior associated with repeated transients only as $\epsilon$ increases. Consequently the modulation period $2\pi/\omega_3$ is infinite when the repeated transients first appear, but drops rapidly with increasing $\epsilon$, as observed in figure 4 of Kolodner [1993]. When the condition $\rho < 1$ on the eigenvalues at A and B holds the resulting repeated transients are expected to be chaotic. The repeated transients persist only over a narrow interval of $\epsilon$ close to onset, of order $10^{-3}$ (see figs. 5.23,5.26), which compares well with the observed range $2 \times 10^{-3}$ for $\Gamma = 16.75$ [Kolodner, 1993], and give way to large amplitude blinking states in a (slightly) hysteretic transition when $\epsilon \approx 10^{-3}$. Our calculations suggest that dispersive effects are not of fundamental importance in this behavior (in contrast to Kolodner's conjecture), but that the primary chevron state must bifurcate subcritically. In particular we have been unable to identify any fundamental role of the local wavenumber changes observed in the experiments.

- Stable large amplitude blinking states set in via a (typically hysteretic) bifurcation from the repeated transient state. This bifurcation eliminates the slowest frequency from the time trace.

- In the experiments the blinking states become irregular with increasing $\epsilon$ and the blinking period gradually increases, until an abrupt transition to a localized traveling wave state attached to one boundary (or to steady overturning convection) takes place. The observed chaotic blinking is likely associated with the global bifurcation studied by Dangelmayr, Knobloch, and Wegelin [1991], and Knobloch [1996].

- Although the chaotic asymmetric blinking states are easily confused with the repeated transients because of their somewhat similar appearance, cf. Kolodner [1993], such states develop from symmetric blinking states with increasing $\epsilon$ and do so either via spatial symmetry-breaking followed by period doubling, or via a further Hopf bifurcation.

- Regular blinking states are observed near onset only for aspect ratios differing roughly by unity, as in the experiments. Our simulations suggest (fig. 5.20) that these special aspect ratios are in fact nothing but the mode interaction points (compare fig. 4.7(a) with fig. 5.20). The location of these points depends quite sensitively on
the system parameters and in particular on the additional dissipation due to the neglected no-slip walls in the third (transverse) direction. Consequently, differences between the experimental results and our calculations may be primarily due to differences in the location of these points. For example, Kolodner finds that blinking states persist down to small amplitudes for $\Gamma = 16.63$ and $\Gamma = 17.63$ when the (dimensionless) width $\Gamma_y = 3.0$, and for $\Gamma = 16.25$ and $\Gamma = 17.25$ when $\Gamma_y = 4.9$. In contrast our strictly two-dimensional calculations ($\Gamma_y = \infty$) show that blinking states are most easily found near $\Gamma \approx 16.8$. In contrast to the experiments [Kolodner, 1993] we do not find that repeated transients predominate at all aspect ratios away from the mode interaction points. For example, at $\Gamma \approx 16.25$ we found subcritical stable chevrons that bifurcate into blinking states with increasing $\epsilon$ (fig. 5.22).

- The numerical simulations support much of the theory summarized in section 2 as to the origin of the different types of states, and provide essential information supporting the interpretation of the repeated transients as a three-frequency state. The simulations also confirm the significance of the parameter $\Gamma \bmod \pi/k_c$ proposed in Dangelmayr, Knobloch, and Wegelin [1991] and confirmed so dramatically in Kolodner’s experiments [Kolodner, 1993]. This parameter reflects the fact that when $\Gamma$ changes by $\pi/k_c$, where $k_c$ is the “wavenumber” of the waves, an additional roll fits into the container, thereby changing the parity of the most unstable state. Detailed calculations [Batiste, Mercader, Net, and Knobloch, 1999] indicate that the wavenumber at onset is in fact remarkably uniform across the cell (cf. Kolodner [1993]), despite the variation in amplitude, and that $k \approx \pi$. Thus when the neutral curves of both modes cross, the system feels comfortable with both, and oscillates regularly even in the nonlinear regime. This is not so for other aspect ratios for which one mode is preferred, and when that loses stability at finite amplitude to a symmetry-breaking perturbation the competing state does not fit well into the container and the system oscillates irregularly.

These results and the accompanying interpretation account quantitatively for almost all of the observations of Kolodner [1993] on water-ethanol mixtures, and suggest related experiments on $^3$He-$^4$He mixtures. Recent improvements in experimental technique make the latter viable [Woodcraft, Lucas, Matley, and Wong, 1999]. In particular the quantitative success of the numerical simulations confirms that the restriction to two dimensions is not fatal, and helps in identification of the different dynamical states of the system. It is our view that the remaining quantitative discrepancies can all be attributed to the sensitive dependence of the mode interaction point on the width of the container, because of its effect on dissipative processes in the cell.
The results reported here indicate that for sufficiently small $|\epsilon|$ the sidewalls necessarily exert a critical influence on the dynamics of the system. This is to be expected since for such $\epsilon$ the behavior of the system is dominated by one (or at most two) unstable modes of the system, whose spatial structure is determined by the lateral boundary conditions. Order of magnitude estimates [Dangelmayr, Knobloch, and Wegelin, 1991; Renardy, 1999] suggest that this will be the case whenever $|\epsilon|\Gamma^2 \lesssim 1$, i.e., with increasing $|\epsilon|$ the sidewall influence becomes smaller, and indeed one may reach the situation in which the collapse described by the subcritical Ginzburg-Landau equation on an unbounded domain becomes a more and more appropriate description of the dynamics [Kaplan, Kuznetsov, and Steinberg, 1994]. The experiments of Kaplan, Kuznetsov, and Steinberg [1994] suggest that this is in fact so once $|\epsilon|\Gamma^2 \gtrsim 10$. The results of Niemela, Ahlers, and Cannell [1990], performed for slightly larger $|S|$ and subcritical values of $\epsilon$, can likewise be interpreted as showing that $|\epsilon|\Gamma^2 \approx 5$ describes the transition between these two regimes. However, the water-ethanol simulations reported here all satisfy the condition $|\epsilon|\Gamma^2 \lesssim 1$, and hence are always dominated by the sidewalls.

The model put forward in section 6 suggests that equations (2.10) can only describe repeated transients if the primary bifurcation to the chevron state is subcritical, i.e., if $A_R > 0$, and if fifth order terms are included. However, as discussed in detail by Moehlis and Knobloch [1998, 2000] the cubic truncation may suffice to describe the type of bursts observed by Sullivan and Ahlers [1988] in $^3$He-$^4$He mixtures. These experiments are performed in containers extended in one dimension, with $\Gamma = 34$, and the bursts located at $\epsilon = 3 \times 10^{-4} < \Gamma^{-2}$, i.e., the reported bursts should be amenable to a finite-dimensional description. Although these bursts require the presence of at least one subcritical branch fifth order terms are not necessarily required to provide saturation. Unfortunately, despite much effort we have been unable to reproduce these bursts in our simulations.

Acknowledgments: Preparation of this article was supported by EPSRC grant GR/R52879/01 and the National Science Foundation, under grant DMS-0072444. Computer time was provided by CEPBA. We also thank I. Mercader and M. Net for assistance.

References


3. Oscillatory Binary Fluid Convection


