Periodic and localized states in natural doubly diffusive convection

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Abstract

Numerical continuation is used to follow branches of steady doubly diffusive convection in a vertical slot driven by imposed horizontal temperature and concentration gradients. No-slip boundary conditions are used on the lateral walls; periodic boundary conditions with large spatial period are used in the vertical direction. A variety of different states, both spatially periodic and spatially localized, are identified and the profusion of the resulting solution branches is linked to a phenomenon known as homoclinic snaking.

Experiments with parallel horizontal gradients have been carried out by Kamotani et al. [23], Jiang et al. [21,22], Lee et al. [26,27] and by Han and Kuehn [18]. In these experiments the Prandtl number (\(Pr \equiv \nu/\kappa\), where \(\nu\) is the kinematic viscosity and \(\kappa\) the thermal diffusivity) is of order 10 and the Lewis number (\(Le \equiv \kappa/D\), where \(D\) is the concentration diffusivity) is of order 100–500. The value of \(N\) is adjusted to a prescribed value by changing the ratio \(\Delta C/\Delta T\) and the experiments cover the range \(-100 < N < 40\). In many cases the fluid container is extended in the direction transverse to the imposed gradients, and the observations are then in qualitative agreement with the results of two-dimensional numerical simulations. Even with this simplification the simulations indicate that in containers of order 1 aspect ratio, both single-roll and multicellular flows can be observed depending on the values of \(N\) and \(Le\) [18]. For larger Lewis numbers the formation of steady or unsteady solutal boundary layers appears to be responsible for the increased difference between the experiments and the results of two-dimensional simulations.

The behavior in the cooperating case resembles that familiar from natural convection [3,4,17]. The opposing case is quite different, however. In the present work we focus on the case \(N = -1\) that captures the essence of this case, and nearby

1. Introduction

Doubly diffusive convection, that is, convection driven by a combination of concentration and temperature gradients, has been the subject of much study. The system is known to display a wealth of dynamical behavior whose properties depend both on the direction and the magnitude of the initial (or imposed) gradients [42,43]. The earliest work was motivated by oceanographic applications [34] and focused on situations in which the gradients are parallel to the buoyancy force. In contrast, in solidification, convection is frequently induced by gradients perpendicular to the buoyancy force [44]. The latter is the situation of interest in the present paper. We measure the relative importance of the temperature and concentration gradients using the buoyancy ratio \(N = \rho_C \Delta C/\rho_T \Delta T\), where \(\rho_T < 0, \rho_C > 0\) are the thermal and solutal ‘expansion’ coefficients, and \(\Delta T, \Delta C\) are the differences in temperature and concentration imposed across the system. When the imposed gradients are parallel, the two buoyancy forces are said to be cooperating if \(N > 0\), and opposing if \(N < 0\).
values. When \( N = -1 \) there exists a pure conduction state with linear profiles of temperature and concentration, and linear stability analysis can be used to identify a threshold for the first instability. Bifurcation theory can then be used to characterize the states that develop. In two dimensions such a study was performed numerically for a variety of aspect ratios by Gobin and Bennacer [16], Ghorayeb and Mojtabi [15], Bergeon et al. [6] and Xin et al. [46], and extended to three dimensions by Sezai and Mohamad [35] and Bergeon and Knobloch [7]. Bardan et al. [1] have employed weakly nonlinear analysis to discuss the effects of small departures from \( N = -1 \), and used the results to understand much of the structure observed in numerically generated bifurcation diagrams for other values of \( N \).

The present paper is motivated by the observation by Ghorayeb and Mojtabi [15], see also [14], of an unusual single-cell state (labeled Type 1 in their Fig. 13) in a numerical study of the \( N = -1 \) case in a vertically extended container (aspect ratio \( 7 \)). This cell is located at the center of the container, away from either end, and hence takes the form of what we call below a spatially localized state. To the authors’ knowledge the origin of this intriguing state has not hitherto been elucidated. We show here that the presence of this state is intimately related to the subcriticality of the primary instability of the conduction state (also demonstrated in [15]). To do so we examine the nature of the solutions in a vertical slot of height that is much larger than the width. We impose no-slip boundary conditions only along the sidewalls, and use periodic boundary conditions in the vertical direction. We choose this period, hereafter \( \Gamma \), to be large in order to accommodate a substantial number of cells, but restrict attention to two-dimensional flows. Despite this simplification we uncover unexpected richness in the variety of possible steady states: in addition to the branch of periodic cells, there are several secondary branches that ‘snake’ back and forth in the bifurcation diagram. By examining the limit of an infinitely long slot we show that these branches correspond to branches of spatially localized states. There are two types of such states: those that approach the conduction states at large distances, and those that resemble ‘holes’ in an otherwise periodic pattern of cells. We show that the former bifurcate from the conduction state at the same Grashof number as the spatially periodic states, while the latter bifurcate from the periodic states near the saddle-node at which these states turn around towards larger Grashof numbers. We show how these bifurcations are modified when the imposed spatial period is finite, and use the results to make sense of the complexity of the bifurcation diagram obtained for \( \Gamma = 7.5 \approx 3 \lambda_c \), where \( \lambda_c \approx 2.5 \) is the critical wavelength of the instability at onset.

The paper is organized as follows. In the next two sections we introduce the governing equations and describe the numerical method used. In Section 4 we present and discuss our results. In Section 5 we discuss what we expect to occur in an infinite slot, and confirm certain aspects of our predictions using spatial dynamics. We also discuss the expected effects of finite spatial period on the spatially localized states predicted by spatial dynamics, and explain how these are related to the results of Section 4. The paper concludes with a brief summary.

2. Governing equations

We consider a non-reaktive binary fluid mixture confined in a two-dimensional vertical container with two opposite sidewalls, at \( x = 0, \ell \), maintained at prescribed (and unequal) temperatures and concentrations. We denote the spatial period in the \( z \) direction by \( H \), and define the dimensionless aspect ratio to be \( \Gamma = H/\ell \) (see Fig. 1). We suppose that the wall \( x = 0 \) is maintained at a constant temperature \( T_r^* + \Delta T \) and a concentration \( C_r^* + \Delta C \), while the wall at \( x = \ell \) is maintained at \( T_r^* \) and \( C_r^* \), and assume that \( \Delta T > 0, \Delta C > 0 \). We use the Boussinesq approximation with the fluid density \( \rho \) given by

\[
\rho(T^*, C^*) = \rho_0 + \rho_\ell (T^* - T_r^*) + \rho_\ell (C^* - C_r^*),
\]

where \( T^* \) and \( C^* \) are, respectively, the temperature and concentration, and \( \rho_\ell < 0 \) and \( \rho_\ell > 0 \) denote the thermal and solutal ‘expansion’ coefficients at the reference temperature and concentration. In the following we use the symbols \( T \) and \( C \) to refer to the quantities \( T^* - T_r^* \) and \( C^* - C_r^* \) nondimensionalized using \( \Delta T \) and \( \Delta C \), respectively, and use \( \ell, \ell^2/\nu, \nu/\ell \) as units of length, time and velocity, where \( \nu \) is the kinematic viscosity.

The resulting dimensionless governing equations read

\[
\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \nabla^2 \mathbf{u} + Gr(T + NC)e_z, \tag{2}
\]

\[
0 = \nabla \cdot \mathbf{u}, \tag{3}
\]

\[
\frac{\partial T}{\partial t} = -(\mathbf{u} \cdot \nabla) T + \frac{1}{Pr} \nabla^2 T, \tag{4}
\]

\[
\frac{\partial C}{\partial t} = -(\mathbf{u} \cdot \nabla) C + \frac{1}{Sc} \nabla^2 C, \tag{5}
\]

where \( \mathbf{u} \equiv (u, w) \) and \( \mathbf{V} \equiv (\partial_x, \partial_z) \) in \((x, z)\) coordinates, with \( x \) in the horizontal direction and \( z \) in the vertical direction. The Prandtl number \( Pr \), the Schmidt number \( Sc \), the Grashof
number $Gr$ and the buoyancy ratio $N$ are defined by

$$Pr = \frac{v}{\kappa}, \quad Sc = \frac{v}{D}, \quad Gr = \frac{g\rho C}{\rho_0 T^3}.$$  

$$N = \frac{\rho C \Delta C}{\rho_0 \Delta T^3},$$  

where $\kappa$ and $D$ are the thermal and solutal diffusivities, and $g$ is the gravitational acceleration. In terms of these quantities the Rayleigh number $Ra \equiv Gr Pr$ and the Lewis number $Le \equiv Sc/Pr$. The boundary conditions read

at $x = 0 : u = w = T - 1 = C - 1 = 0$,

at $x = 1 : u = w = T = C = 0$,  

(7)

together with periodic boundary conditions with period $\Gamma$ in the $z$ direction. The results that follow are computed for $Sc = 11$ and $Pr = 1$.

In writing these equations we have neglected two cross-diffusion effects that lead to the segregation of the heavier and lighter components of the fluid in an imposed temperature gradient (the Soret effect), or the development of a temperature gradient in a system with an imposed concentration gradient (the Dufour effect). Both effects are present in fluids, although in liquids generally only the former is significant. However, except in special circumstances the additional gradients produced by either effect will be dwarfed by the externally imposed gradients assumed here.

Owing to the periodic boundary conditions in the vertical, the above problem, as posed, is invariant under translations in $z$. In addition, the equations are invariant under a rotation by $180^\circ$, viz.,

$$\Delta : (x, z) \rightarrow (1-x, -z),$$

$$\Delta (u, w, \Theta, \Sigma) \rightarrow -(u, w, \Theta, \Sigma),$$  

(8)

where we have written $T = 1 - x + \Theta, C = 1 - x + \Sigma$. Here $z = 0$ is an arbitrary location along the slot. Together these two operations generate the symmetry group $O(2)$ of rotations and reflections of a circle. In these circumstances general theory [11] guarantees the presence of $\Delta$-symmetric solutions satisfying

$$(u, w, \Theta, \Sigma)(1-x, -z) = -(u, w, \Theta, \Sigma)(x, z)$$  

(9)

relative to a suitable origin in $z$. There are two basic types of such states: periodic states with wavelength $\lambda_c$ and periodic states with wavelength $\Gamma$. For simplicity, in the following we refer to the former as periodic states and the latter as localized states. Since we focus here on solutions with spatial period $\Gamma$, translations by $\Gamma/2$ are of importance. For periodic states with an even number of wavelengths within the period $\Gamma$ this translation acts like an identity. Steady state bifurcations from such states that break this symmetry will necessarily be pitchforks. On the other hand, steady state bifurcations from periodic states without this additional symmetry (i.e., states with an odd number of wavelengths per period) will be generic bifurcations, and hence either saddle-node or transcritical bifurcations. In the following we shall encounter both types of behavior.

It should be noted that the symmetry $\Delta$ remains a symmetry of the problem even in a finite container, provided that the boundary conditions at the top and bottom are identical. This is so even in three spatial dimensions. See Ref. [7] for further discussion. Thus $\Delta$-symmetric states are of particular importance in this type of problem. However, secondary bifurcations from these states will be affected qualitatively unless the boundary conditions at the top and bottom are of Neumann type (stress free, no-vertical flux).

In this paper we fix $N = -1$. The problem then has the trivial solution $u \equiv 0$, $T(x, z) = C(x, z) = 1 - x$, and this solution is linearly stable up to a critical Grashof number $Gr_{P_1} > 0$, computed in Ref. [46] for both elongated cavities and for an infinite slot. At this critical parameter value the conduction state loses stability to a $\Delta$-symmetric state with wavelength $\lambda_c$. In the following we choose $\Gamma = 7.5 \approx 3\lambda_c$, a value close to that employed in the original paper by Ghorayeb and Mojtabi [15], but chosen to admit an integer number of critical wavelengths. Additional primary bifurcations occur as $Gr$ is increased above $Gr_{P_1}$, as discussed in Ref. [7], and we will be interested in branches bifurcating at $Gr_{P_1}$ and $Gr_{P_2}$ as well. All are pitchforks of revolution, a consequence of the periodic boundary conditions in $z$.

3. Numerical method

In order to compute steady solutions of the above problem as a function of the Grashof number we use a numerical continuation method based on a Newton solver for the time-independent version of Eqs. (2)–(5) with the boundary conditions (7). The implementation of the method follows that of Tuckerman [41] and Mamun and Tuckerman [29]. Its application to the three-dimensional version of the situation considered here is described in detail elsewhere [7]. The main difference lies in the discretization of the equations. Here, we use a spectral element method in which the domain $[0, 1] \times [0, \Gamma]$ is decomposed into $N_e$ macroelements of size $[0, 1] \times [i \Gamma/N_e, (i + 1) \Gamma/N_e]$, where $N_e$ is the number of spectral elements and $i \in [0, \ldots, N_e - 1]$. In each element, the fields are approximated by a high order interpolant through the Gauss–Lobatto–Legendre points [13]. The Newton solver requires a first-order time integration scheme for the equations in conservation form; we use the scheme proposed by Karniadakis et al. [24] in which the diffusive linear part of the equations is treated implicitly. Each time step therefore requires the inversion of four Helmholtz problems. The discrete version of the weak form of the Helmholtz problem in the whole domain is first transformed into $N_e + 1$ one-dimensional problems in the $z$ direction after a partial diagonalization of the operator, a procedure made possible by the tensorization of the operator in the $x$ direction [28]. Each of these $N_e + 1$ matrix problems of size $N_e N_z \times N_e N_z$ is solved using a Schur decomposition method by separating the matrix elements acting on the nodes inside each element from those acting on nodes located along the interface between two adjacent elements [12,31,33,9]. The Schur matrices that solve each problem on the interfacial points are of size $N_e \times N_e$ and are
inverted using a direct method. The variational formulation of the problem ensures the continuity of the unknowns and their first derivatives in the $z$ direction across the interface between adjacent elements. The periodicity in the $z$ direction is enforced by taking the boundaries $z = 0$ and $z = Γ$ to be the unique interface between the first and last spectral element.

In the present work we study only steady states with $Δ$-symmetry, i.e., we impose the requirement (9) on all solutions.

4. Results

In this section we present the bulk of our numerical results. All figures show $w(P)$ as a function of $Gr$, where $P$ is the (grid) point $(x, z) = (0.28968, 0.42995)$. We refer to solutions in terms of an integer $n$ specifying the period $Γ/n$ of the solution. Thus for primary branches, $n$ specifies the number of pairs of rolls within the imposed spatial period. In general the rolls within a pair are asymmetrical, with one sense of circulation stronger than the opposite. On large amplitude branches the weaker circulation may be absent altogether. The plots of $w(P)$ distinguish between $Γ/2n$ translates of a periodic solution; thus $Δ$-symmetric solutions that are related by translation will appear as distinct branches.

Fig. 2 summarizes all our results. Solid (dashed) lines indicate stable (unstable) solutions. The solid dots indicate steady state bifurcations other than saddle-node bifurcations which are not explicitly indicated. Open circles indicate Hopf bifurcations, while open triangles show the location where a pair of complex eigenvalues collides on the real axis. The latter are not bifurcations but indicate that the nature of the instability has changed from a growing oscillation to a monotonically growing perturbation. Apart from the identification of competing stable branches (several are shown in Fig. 2) one of our objectives is to elucidate the origin of the profusion of branches seen in the figure, and to shed light on the relation between these branches.

At first sight this appears to be a difficult if not impossible task. It turns out, however, that there is much predictable order in the figure, and the key to this order is provided by the extended system, corresponding to large spatial period $Γ$. In order to appreciate this it is useful to split up the branches into several sets. Fig. 3 shows the primary bifurcation $P_1$. As already mentioned this bifurcation is a pitchfork of revolution, but in our representation it simply looks like a (subcritical) pitchfork. The solutions along the two branches that emerge are related by translation by $λ_c / 2$ (equivalently by $Γ/2$) and hence are the same solutions. The panels at the left show these solutions at finite amplitude and identical $Gr$ value, and reveal that the solution that comes in at $P_1$ is an $n = 3$ state. Since $Sc > Pr$ the clockwise rolls are (slightly) stronger than the counterclockwise rolls; this is a finite amplitude effect. The solutions shown are located just past a secondary transcritical bifurcation at $S_1$ (resp. $S_1'$) that is discussed further below. Since the primary solutions are in fact the same solution, $S_1$ and $S_1'$ represent the same bifurcation, and hence occur at identical $Gr$. As $Gr$ is decreased below $S_1$ (resp. $S_1'$) both branches pass through a saddle-node bifurcation at the same value of $Gr$ (not indicated), beyond which they are once unstable. However, almost immediately both undergo a secondary bifurcation at $S_2$ (resp. $S_2'$) at which both acquire stability. The figure shows the stable solutions that result at $Gr = 800$; the counterclockwise rolls are now absent, and their location between adjacent clockwise cells is taken up by a shear zone. Note that the solutions at $Gr = 800$ are still translates of one another by $Γ/2n$.

Fig. 2. Bifurcation diagram showing the velocity at one point versus the Grashof number. Solid (dashed) lines refer to stable (unstable) steady states. Dark circles indicate pitchfork or transcritical bifurcation points, while open circles show Hopf bifurcation points and triangles collisions of complex eigenvalues on the real axis. Saddle-node bifurcations are not indicated. The diagram shows five solution branches in addition to the conduction state $w(P) = 0$. The spatial period $Γ = 7.5$, while $Pr = 1$ and $Sc = 11$. The resolution is $N_e = 4$ with $N_x = 16$ and $N_z = 14$.

Fig. 3. The branch of periodic states bifurcating from the conduction state at $P_1$, showing the velocity at one point versus the Grashof number. Solid (dashed) lines refer to stable (unstable) steady states. Dark circles indicate pitchfork or transcritical bifurcation points, while open circles show Hopf bifurcation points and triangles collisions of complex eigenvalues on the real axis. Saddle-node bifurcations are not indicated. Stability is indicated by two numbers, the first being the number of real unstable eigenvalues, and the second the number of pairs of complex eigenvalues with a positive real part. The secondary bifurcations at $S_2$, $S_2'$ to hole-like states are preceded by nearby saddle-node bifurcations (not indicated). The spatial period $Γ = 7.5$, while $Pr = 1$ and $Sc = 11$. For these parameters $Gr P_1 = 650$, $Gr S_1 = 632.3$ and $Gr S_2 = 517$. The resolution is $N_e = 4$ with $N_x = 16$ and $N_z = 14$. 
In Fig. 4 we examine the secondary branches that emerge from the points $S_2$ (resp. $S'_2$) above the saddle-node bifurcations. These bifurcations are (slightly) transcritical and produce a pair of branches of $n = 1$ solutions, one of which turns around almost immediately, much as occurs near $S_1$ (see below). The middle panels in the figure show the fully developed solutions arising from the bifurcation at $S_2$. Both share the even symmetry of the primary branch. However, along the upper branch the $n = 1$ mode suppresses the two central clockwise rolls, while strengthening the remaining roll. In contrast, along the lower branch the two central rolls are strengthened, while the remaining roll is weakened. In the following we refer to the resulting solutions as ‘hole-like’. Evidently these two branches are now quite distinct, and indeed the upper branch encounters a Hopf bifurcation at $H_1$, while the lower one does not. Analogous transitions occur along the branches appearing from $S'_2$. Here the solutions on the lower branch are $\Gamma/2$ translates of the solutions on the upper branch arising from $S_2$ and vice versa. As a result the lower branch encounters a Hopf bifurcation at $H'_1$, at the same Grashof number as $H_1$. Since they are related, these two branches can come together again, and indeed do so in another transcritical bifurcation, labeled $S_3$ in the figure. In contrast, the other branches arising from $S_2$ and $S'_2$ continue to larger Grashof numbers, without suffering additional bifurcations.

Fig. 5 describes the fate of the solution branches emanating from the secondary transcritical bifurcations at $S_1$ (resp. $S'_1$), with Figs. 6 and 7 showing enlargements. We describe here what happens along the pair of branches emanating from $S'_1$. Following the upper branch from $S'_1$ the branch first undergoes a saddle-node bifurcation at the right (Fig. 7) before acquiring stability at a saddle-node bifurcation on the left (Fig. 6); the branch then loses stability at a second saddle-node bifurcation on the right, and this is followed by the acquisition of stability at $S_3$ (see below), and its almost immediate loss at a saddle-node bifurcation. The branch then continues towards larger amplitude in a crescent shape before abruptly turning around towards smaller $Gr$ and acquiring stability again. The branch then loses stability at a saddle-node and ends at $S_1$ after a further saddle-node bifurcation. In contrast, the lower branch starting at $S'_1$ acquires stability at a saddle-node on the left, and then continues towards larger values of $Gr$ until it loses stability in a Hopf bifurcation at $H'_2$. The lower branch at $S_1$ behaves in a similar fashion, ultimately losing stability at $H_2$.
We emphasize two aspects of the above behavior. First, there are several bifurcations (the saddle-nodes and the Hopf bifurcations) that are common to the branches emanating from $S'_1$ and the corresponding branches emanating from $S_1$, and that occur at identical values of $Gr$. This behavior indicates that the solutions on these branches are $n = 1$ states that are related by $\Gamma'/2$ translation, and that the associated eigenfunctions respect the imposed $\Delta$-symmetry. In addition, we see that the $n = 3$ solution at $S'_1$ evolves along the upper branch emanating from $S'_1$ into a single clockwise cell in the center of the domain by the time it first acquires stability at a left saddle-node (Fig. 6). We also see a sign of a weak second cell by the time the branch loses stability again at a right saddle-node (Fig. 6), and that this two-cell structure is fully developed by the time the branch reaches $S_3$; indeed, at $S_3$ the solution is an $n = 2$ periodic state. The process then repeats in reverse until the branch returns to $S_1$. The above conclusion suggests that (a) the branch that emanates from $S'_1$ gains and loses stability repeatedly at saddle-node bifurcations, and that these bifurcations are associated with the repeated additions of cells as a single-cell localized state broadens in spatial extent. As discussed in much greater detail in the following section this type of behavior provides the basic key required for the understanding of the complexity seen in Fig. 2.

We now turn to Fig. 8 (see also Fig. 9) and the branches that emanate from the second primary bifurcation at $P_2$. The figure shows that the two branches that arise at this point correspond to an $n = 4$ state; once again the solutions on these branches are related by translation by $\lambda/2$. Note, however, that this time the stability properties of these branches are not identical. This is a consequence of the fact that we impose $\Delta$-symmetry on both the basic states and the allowed perturbations, and that $n$ is even; only if $\Delta$-symmetry breaking perturbations are admitted as well will the two branches emanating from $P_2$ have identical stability properties. With this proviso the secondary bifurcation at $S_4$ on the upper branch (solutions with maxima in the center of the domain) implies the presence of a secondary bifurcation $S'_4$ (not shown) at the same $Gr$ value on the lower branch (solutions with minima in the center of the domain). Thus the two solutions shown at $Gr = 800$ do in fact have identical stability properties, and both are twice unstable. Moreover, the saddle-node bifurcations on the two branches coincide; this is a consequence of the fact that a saddle-node bifurcation is a bifurcation in amplitude, and so is unaffected by any symmetries imposed on the allowed perturbations.

Fig. 10 shows the branches emanating from the third primary instability, at $P_3$. The figure shows that this branch is an $n = 2$ branch. Once again the saddle-node bifurcations on the upper and lower portion of the branch line up, indicating that the solutions on these branches are $\lambda/2$ translates of one another. However, since $n$ is even the stability properties of these translates will in general be different. Indeed, the upper branch emanating from $P_3$ undergoes a secondary bifurcation at $S_5$, while there is no corresponding bifurcation on the lower branch. At larger amplitude the upper branch undergoes a pair of saddle-node bifurcations, before acquiring stability at a Hopf bifurcation labeled $H_3$. After an interval of stability this branch loses stability at $S_5$; a second eigenvalue becomes unstable almost immediately at a saddle-node bifurcation, followed by a third at $S_4$. After four further saddle-node bifurcations the branch reconnects to the primary bifurcation point $P_3$. The
Fig. 8. The branch of periodic states bifurcating from the conduction state at $P_2$, showing the velocity at one point versus the Grashof number. Solid (dashed) lines refer to stable (unstable) steady states. Dark circles indicate pitchfork or transcritical bifurcation points, while open circles show Hopf bifurcation points and triangles collisions of complex eigenvalues on the real axis. Saddle-node bifurcations are not indicated. Stability is indicated by two numbers, the first being the number of real unstable eigenvalues and the second the number of pairs of complex eigenvalues with positive real part. The spatial period $\Gamma = 7.5$, while $Pr = 1$ and $Sc = 11$. For these parameters $Gr_{S_2} = 698.8$ and $Gr_{P_2} = 700$. The resolution is $N_e = 4$ with $N_x = 16$ and $N_z = 14$.

Fig. 9. Detail of Fig. 8.

bifurcation points $S_3$ and $S_4$ were encountered in Figs. 5 and 9: $S_3$ is a pitchfork on the $n = 2$ branch in Fig. 10, while $S_4$ is a pitchfork on the $n = 4$ branch in Fig. 9. The former results in an $n = 1$ contribution to an $n = 2$ state, while the latter is responsible for an $n = 2$ contribution to an $n = 4$ state.

Fig. 10. The branch of periodic states bifurcating from the conduction state at $P_3$, showing the velocity at one point versus the Grashof number. Solid (dashed) lines refer to stable (unstable) steady states. Dark circles indicate pitchfork or transcritical bifurcation points, while open circles show Hopf bifurcation points and triangles collisions of complex eigenvalues on the real axis. Saddle-node bifurcations are not indicated. Stability is indicated by two numbers, the first being the number of real unstable eigenvalues, and the second the number of pairs of complex eigenvalues with positive real part. The spatial period $\Gamma = 7.5$, while $Pr = 1$ and $Sc = 11$. For these parameters $Gr_{P_3} = 731$ and $Gr_{H_3} = 644.1$. The resolution is $N_e = 4$ with $N_x = 16$ and $N_z = 14$.

Thus both bifurcations are subharmonic bifurcations and hence they are mathematically analogous. Once again $\Delta$-symmetry breaking perturbations must be admitted to restore identical stability properties to the two branches emanating from $P_3$.

It will have been observed that the secondary bifurcations identified in the figures all fall close to either saddle-node bifurcations or the primary bifurcations. Table 1 shows, for example, that the bifurcation $S_1$ moves closer and closer to the primary bifurcation at $P_1$ as the spatial period $\Gamma$ increases. Likewise $S_4$ moves towards $P_2$. Moreover, $S_1$, $S_2$ and $S_3$ all occur very close to saddle-nodes and the distance from these saddle-node bifurcations appears to decrease rapidly as the period $\Gamma$ increases. As we show in the next section, this behavior is not an accident, and is a finite period manifestation of bifurcations that are fundamentally bifurcations on an infinite domain: bifurcations to spatially localized states.

5. Theoretical interpretation

Recent work has led to the realization that in extended systems with a subcritical primary instability spatially localized states will bifurcate from the trivial state at the same parameter value as the much more frequently studied spatially periodic
states (see [2,8] for recent references). Although these localized states are initially unstable, they may undergo a transition to so-called ‘snaking’ as the parameter decreases (Fig. 11) and acquire stability. The snaking region is a region of parameter values that straddles (in variational systems) the Maxwell point at which both the trivial (conduction) state and a large amplitude spatially periodic state have identical energies. In many problems, including the one described here, the large amplitude spatially periodic state is present because the periodic solutions that bifurcate subcritically from the conduction state turn around at a saddle-node bifurcation. In nonvariational problems, such as the present one, an analogue of a Maxwell point is still present, and now corresponds to the formation of a heteroclinic connection from the conduction state to the periodic state; in systems that are spatially reversible the reversibility guarantees the simultaneous formation of a heteroclinic connection from the periodic state back to the conduction state, i.e., the formation of a heteroclinic cycle. The presence of such a point must be determined numerically, but may be associated with snaking nonetheless. Indeed theory shows that generically the stable and unstable manifolds of the conduction state will intersect transversally, resulting in a snaking region [10,45]. This terminology refers to the back and forth oscillations in parameter space of branches of spatially localized states that take place as the localized state grows in width by the addition of rolls at both ends; after an infinite number of back and forth oscillations the branch of localized states terminates on a branch of periodic states. Thus the snaking region is characterized by multiple stable localized states that gain and lose stability as one proceeds past successive saddle-node bifurcations. The snaking itself is a consequence of spatial locking between the fronts bounding the localized state and the spatially periodic state within it.

In the problem at hand the primary bifurcation at \( P_1 \) is a subcritical bifurcation that leads to a spatially periodic pattern. The resulting branch is initially unstable (see Fig. 3), but turns around towards larger Grashof numbers at a saddle-node bifurcation, and acquires stability shortly thereafter at the bifurcation labeled \( S_2 \). In addition, the symmetry \( \Delta \) makes the problem \( \Delta \)-reversible, i.e., the operation \( z \rightarrow -z \) acts as \(-1\). Thus the present problem satisfies the conditions that are usually associated with the presence of spatially localized states and in particular of homoclinic snaking.

To examine the above suggestion at greater depth we employ the technique of spatial dynamics to obtain insight into the origin of spatially localized states on the real line \(-\infty < z < \infty\). Specifically, we seek time-independent solutions that evolve

\[
|u(z)|
\]

with \( z \) from the conduction state \( u = w = \Theta = \Sigma = 0 \) at large negative \( z \) into a convecting state, and then back into the conduction state as \( z \rightarrow \infty \), i.e., we seek localized states as homoclinic orbits connecting the conduction state to itself. Whether such orbits are possible depends in part on the stability properties of the conduction state: spatial eigenvalues with positive real part indicate that a nontrivial state can grow from \( z = -\infty \), while eigenvalues with negative real part indicate that such a state may return, under appropriate conditions, back to the trivial state as \( z \rightarrow \infty \). The spectrum of the linearization about the trivial state is influenced by spatial symmetries of the system, here \( \Delta \)-reversibility. As a consequence the linearized problem is invariant under \( z \rightarrow -z \), and the bifurcations that are encountered as \( Gr \) varies are nongeneric.

With the above background we turn to the study of spatially localized states. The first class of localized solutions of interest are small amplitude stationary states biaxsymptotic to the conduction state that exist, for example, near \( Gr = Gr_{P_1} \). Explicit solution of the linearized problem near \( Gr = Gr_{P_1} \) shows that the four leading spatial eigenvalues form the quartet \( \pm iq_0 \pm O(\sqrt{r}) \) when \( r \equiv Gr - Gr_{P_1} < 0 \), while for \( r > 0 \) they are all imaginary: \( \pm iq_0 \pm O(\sqrt{r}) \). Here \( q_0 \) is the wavenumber of the primary instability selected at \( Gr = Gr_{P_1} \). It follows that for \( r < 0 \) the conduction state is hyperbolic with two stable eigenvalues and two unstable eigenvalues. In contrast, for \( r > 0 \) all the eigenvalues lie on the imaginary axis and the conduction state is not hyperbolic. In the latter case exponentially localized states are not possible. At \( r = 0 \) there is a pair of imaginary eigenvalues \( \pm iq_0 \) of double multiplicity. The bifurcation at \( r = 0 \) is thus a Hopf bifurcation in a reversible system with 1:1 resonance [20], and an analysis of a particular degeneracy in the normal form for this bifurcation establishes the presence of an exponentially thin snaking region
emerging from the associated codimension 2 point [45]. This snaking region broadens into the observed region as one moves away from this codimension two point.

In the following we establish numerically the presence of the above Hopf bifurcation in the present problem. We do so by examining the spatial eigenvalues $\lambda$ of the conduction state near the primary instability to spatially periodic states. Solutions of Eqs. (2)–(5), linearized about $u = w = \Theta = \Sigma = 0$, of the form $f(x) \exp \lambda z$ satisfy

\begin{align}
(\partial_x^2 + \lambda^2) u &= \partial_x p, \quad (\partial_x^2 + \lambda^2) w + Gr(\Theta - \Sigma) = \lambda p, \\
\partial_x u + \lambda w &= 0, \\
\partial_x^2 \Theta &= -Pr u, \quad (\partial_x^2 + \lambda^2) \Sigma = -Sc u,
\end{align}

with $u = w = \Theta = \Sigma = 0$ on $x = 0$. To solve this problem the unknowns are expanded in Chebyshev polynomials through order $N$ (typically $N = 40$). The linear system is solved for fixed $\lambda = \lambda_r + i\lambda_i$. The minimum of the curve $Gr = Gr(\lambda_r = 0, \lambda_i = q_c)$ defines the critical Grashof number, $Gr \equiv Gr_{p1} = 650.9034$, together with the corresponding critical wavenumber $\lambda_i \equiv q_c = 2.5318$, confirming the results obtained in [6,15,46]. The computation shows that the eigenvalues $\lambda = \pm iq_c$ have double multiplicity, and that for $Gr < Gr_{p1}$ these eigenvalues split, forming a quartet of complex eigenvalues $\lambda$ with small real parts (Fig. 12). In addition there are other eigenvalues bounded away from the imaginary axis.

The eigenvalues with the smallest real part are the leading spatial eigenvalues of the conduction state. Center manifold reduction that preserves the reversibility of the system, followed by appropriate unfolding, can therefore be used to reduce Eqs. (2)–(5) to a time-independent Ginzburg–Landau equation. This equation includes the slow spatial modulation due to the $O(\sqrt{r})$ real part of the leading eigenvalues $\lambda$. To describe the result of such a reduction it is convenient to define a small parameter $\epsilon$ by $r = -\epsilon^2 \mu_2, \mu_2 > 0$, and look for stationary solutions of Eqs. (2)–(5) with the boundary conditions (7) of the form

$$
w(z) = \epsilon \{ A(Z)e^{i\beta z} f(x) + c.c. \} + O(\epsilon^2),
$$

where $Z \equiv \epsilon z$ is a large scale over which the amplitude of the pattern changes, and $f(x)$ represents the (complex) transverse eigenfunction, determined as part of the calculation just described. The center manifold reduction now leads to an equation of the form

$$
A ZZ = \mu_2 A + \beta A |A|^2 + O(\epsilon).
$$

The simplest nontrivial solution of Eq. (13) is the uniform solution

$$
A(Z) = \left( -\frac{\mu_2}{\beta} \right)^{1/2} e^{i\phi} + O(\epsilon),
$$

corresponding to spatially periodic states with period $2\pi/q_c$ near $r = 0$, viz.,

$$
w_{\text{per}}(z) = 2 \left( \frac{r}{\beta} \right)^{1/2} |f(x)| \cos(q_c z + \Phi(x) + \phi) + O(r).
$$

Here $\Phi(x) \equiv \arg f(x)$, and $\phi$ is an arbitrary phase. Our numerical results indicate that these periodic solutions exist for $Gr < Gr_{p1}$. It follows that $r < 0$ and hence that $\beta < 0$.

Other solutions to Eq. (13) can be found in terms of elliptic functions, and localized states correspond to infinite period

Fig. 12. The (a) real and (b) imaginary parts of $\lambda$ as functions of $Gr_c - Gr$, where $Gr_c \equiv Gr(\lambda_r = 0, \lambda_i = q_c) = 650.9034$ is the critical Grashof number and $q_c = 2.5318$. The parameters are $Pr = 1$ and $Sc = 11$. The resolution in the $x$ direction is $N = 40$.

Fig. 13. Approximate critical eigenvectors at (a) $S_1$ and (b) $S_2$. The parameters are $\Gamma = 7.5, Pr = 1$ and $Sc = 11$. The resolution is $N_x = 6, N_z = 16$ and $N_x = 14$. 

\[ \text{Fig. 12. The (a) real and (b) imaginary parts of $\lambda$ as functions of $Gr_c - Gr$, where $Gr_c \equiv Gr(\lambda_r = 0, \lambda_i = q_c) = 650.9034$ is the critical Grashof number and $q_c = 2.5318$. The parameters are $Pr = 1$ and $Sc = 11$. The resolution in the $x$ direction is $N = 40$.} \]
solutions of this type with $A \to 0$ as $Z \to \pm \infty$:

$$A(Z) = \left( \frac{-2\mu_2}{\beta} \right)^{1/2} \text{sech}(\sqrt{\mu_2} Z)e^{i\phi} + O(\varepsilon).$$

This solution corresponds to

$$w_{\text{loc}}(z) = 2 \left( \frac{2r}{\beta} \right)^{1/2} |f(x)| \text{sech}(\sqrt{-r} z) \cos(q_x z + \phi(x) + \phi) + O(r).$$

Like the spatially periodic states this family of solutions is parametrized by $\phi \in S^1$, which controls the phase of the pattern within the cell envelope. Within the asymptotics this phase remains arbitrary: there is no locking between the envelope and the underlying wavetrain at any finite order in $\varepsilon$. However, it is known [5,47] that this is no longer the case once terms beyond all orders are included. These terms prevent arbitrary translations of the pattern within its envelope, and result in slow evolution of $\phi$. This evolution in turn selects specific values of the phase, $\phi = 0, \pi$; these are the only two phases that preserve the symmetry ($z \to -z, w \to -w$) required of $\Delta$-symmetric localized states [8]. It follows that two branches of localized states bifurcate subcritically from $r = 0$, one of which is stable with respect to translations of the envelope relative to the wavetrain and the other unstable. Both are amplitude unstable. Furthermore, near $r = 0$ the two branches are exponentially close but one might expect that with increasing $-r$ the differences will grow and therefore that two branches with distinct norm will emerge from $r = 0$ [8].

The second class of localized states resembles ‘holes’ in an otherwise spatially periodic state. These bifurcate from the saddle-node $Gr_{SN}$ on a branch of spatially periodic states with appropriate period; within our calculation this branch is approximated by the branch emanating from $P_1$, and the saddle-node by $S_2$. To describe the resulting localized states it is convenient to think in terms of the amplitude of the spatially periodic state. This amplitude is uniform in space and passes through a saddle-node bifurcation at $Gr = Gr_{SN}$. We can look at the spatial eigenvalues of this state, just as we did for the conduction state. One finds that at $Gr_{SN}$ this state has a double zero eigenvalue (more precisely, a pair of +1 Floquet multipliers); on the branch above the saddle-node these eigenvalues split into a pair of real eigenvalues with small positive and negative real parts; below the saddle-node the eigenvalues are purely imaginary. Although explicit calculations require the solution of a Floquet problem in space the above discussion suggests that a branch of weakly localized holes with exponentially decaying tails may indeed bifurcate from the uniform states at $Gr_{SN}$. The envelope of these states takes the form

$$A_{\text{hole}}(z) = A_{\text{background}} + a \sqrt{Gr - Gr_{SN}} \text{sech}^2 \left( (Gr - Gr_{SN})^{1/4} \frac{z}{b} \right);$$

(18)

cf. [48]. Here $A_{\text{background}}$ represents the background constant amplitude periodic state, corresponding to either $\phi = 0$ (maximum at cell center) or $\phi = \pi$ (minimum at cell center).

It remains to consider the modification to the above scenario on a periodic domain of large but finite period [19]. On such a domain, homoclinic orbits are replaced by periodic orbits with period $\Gamma$ and spatially localized states as defined in the preceding section are not possible. Instead the branches corresponding to localized states bifurcate from the subcritical branch of periodic states at small but finite amplitude, thereby breaking the symmetry $w \to -w$ between the $\phi = 0, \pi$ states; this amplitude decreases with increasing $\Gamma$, and approaches zero in the limit $\Gamma \to \infty$. Table 1 shows this type of behavior for the secondary bifurcation point $S_1$ in Fig. 3, indicating that the pair of branches that emerge from $S_1$ (see Fig. 5) should be identified with the $\phi = 0, \pi$ branches of localized states present in the infinite system. Fig. 6 confirms that this identification is correct. Fig. 13(a) shows the eigenvector corresponding to $S_1$. The figure reveals that perturbations of the $n = 3$ state in Fig. 3 by this eigenfunction depend on its sign, thereby confirming that $S_1$ must be transcritical, cf. [32]. Moreover the structure of the eigenfunction indicates a gradual transition to a state with a significant $n = 2$ component.

We expect likewise that the hole-like states no longer bifurcate from the saddle-node on a branch of periodic states. This is so for two reasons: the $P_1$ branch is not the ‘right’ periodic branch and in any case the finite period shifts the corresponding bifurcation away from the saddle-node. However, Fig. 3 identifies a transcritical bifurcation at $S_2$ (resp. $S_2'$) that approximates this saddle-node very well; Fig. 13(b) shows the corresponding eigenvector. Once again the structure of the eigenfunction confirms that the bifurcation at $S_2$ must be transcritical and should lead to the formation of either a single cell or a pair of cells depending on the sign of the eigenfunction. Numerical continuation (see Fig. 4) shows that this is indeed so and indicates that, as expected, the hole structure deepens as one moves away from $S_2$, ultimately producing states that resemble the spatially localized structures present along the branches bifurcating from $S_1$ (resp. $S_1'$). Note, however, that in contrast to the corresponding situation for the Swift–Hohenberg equation [19], the secondary branches of localized states bifurcating from $S_1$ and $S_2$ connect, respectively, to $S_3$ and $S_3'$ on the $n = 2$ primary branch, instead of forming a single secondary branch connecting the $n = 3$ primary branch to itself. Moreover, unlike the $n = 3$ primary branch the $n = 2$ branch does not extend to large $Gr$ (see Fig. 10). Both behaviors appear to be a consequence of the relatively small size of the periodic domain $\Gamma$.

We anticipate, therefore, that on a large domain the branch bifurcating from $S_1$ (resp. $S_1'$) will undergo snaking with a larger number of back and forth oscillations. However, in a periodic domain with a finite period snaking cannot go on forever. At some point the localized structure grows to almost fill the domain; at this point the back and forth oscillations of the branch cease and the branch terminates near the saddle-node on the branch of periodic states much as in the Swift–Hohenberg equation [19]. Thus the saddle-node bifurcations on the branch bifurcating from $S_1$ (resp. $S_1'$) are the consequence of the first few turns in a snaking branch before the growth of the localized structure runs into the imposed spatial period.
6. Conclusion

We have conducted a careful linear and nonlinear study of doubly diffusive flows driven by horizontal temperature and concentration gradients in the special case in which their contribution to the overall buoyancy of the fluid vanishes. This case is of particular interest since it admits a primary bifurcation from a conduction state and exhibits nontrivial dynamical behavior quite close to onset. We focused on the linear and nonlinear behavior of this system in a two-dimensional vertical slot with no-slip boundary conditions at the sides and periodic boundary conditions in the vertical with spatial period much larger than the instability wavelength. We have used numerical branch following techniques to uncover a remarkably complex bifurcation diagram (Fig. 2). This bifurcation diagram consists of several interconnected branches, three of which correspond to spatially periodic structures that are born in primary bifurcations while the remaining branches correspond to more-or-less spatially localized structures. The latter are typically created in secondary transcritical bifurcations located close to the primary bifurcations or to the saddle-node bifurcations where the primary branches turn around towards larger values of the Grashof number $Gr$, and undergo a number of back and forth oscillations in $Gr$. These oscillations result in a multiplicity of stable states; these may lose stability at saddle-node bifurcations or at Hopf bifurcations. We have not explored the dynamical behavior arising from the latter.

We have related the behavior summarized above to the behavior expected of an infinite slot. In such a system exponentially localized states can be viewed as homoclinic orbits connecting the conduction state back to itself as $z$ increases from $-\infty$ to $+\infty$. We have shown that even in relatively small periodic domains the bifurcation behavior reflects the behavior of the infinite system, a fact that allowed us to relate the behavior of the computed branches to so-called homoclinic snaking that is familiar from studies of bistable systems on the real line.

All of the states computed here have a reflection symmetry we have called $\Delta$. Such states are invariant under a 180° rotation around the center of the domain. The imposition of this symmetry selects specific solutions from a circle of solutions present as a result of translation invariance in the $z$ direction, and restricts the number of secondary bifurcations. We have done this deliberately because the profusion of branches increases greatly if this symmetry is not imposed, and makes it essentially impossible to understand the resulting bifurcation diagram.

The doubly diffusive configuration studied here has been considered before. Specifically, Xin et al. [46] have computed the threshold $Gr_{p1}$ for the primary instability in both rectangular cavities and the infinite long vertical slot, and showed that in both cases the primary instability is subcritical. In addition, they examined the stability of the resulting spatially periodic states, finding stable solutions above the saddle-node bifurcation at which the periodic states turn around. At larger values of $Gr$ they found a parity breaking bifurcation that breaks the $\Delta$-symmetry of the periodic state and hence leads to a drifting state. Related computations were performed by Shyy and Chen [36] in a square cavity but stability or time dependence was not investigated. Our results reproduce the linear theory of Xin et al. and extend their results to spatial periods in the vertical that are several times larger than the critical period $\lambda_c = 2\pi/q_c$. It is this extension of the problem that has allowed us to identify spatially localized states in this problem.

Closely related phenomena are likely to be present in laterally heated vertical slots filled with a binary mixture with a negative Soret coefficient. This system, studied in [37, 30], is closely related to the doubly diffusive system with imposed temperature and concentration differences in the horizontal [25]. In both systems the primary instability is subcritical, and both are reversible in the sense described here. Thus the main ingredients for the formation of spatially localized structures are present here as well.

Small departures from $N = -1$ can be included as in [1]; these destroy the presence of the conduction state, and render the primary bifurcation imperfect. As a result convection builds up with increasing Grashof number, but the region of bistability present when $N = -1$ remains [30]. Moreover, we expect that the snaking region will be insensitive to such a perturbation, merely shifting in $Gr$. This is because the formation of the pair of heteroclinic connections is a codimension 1 phenomenon in $\Delta$-reversible systems, and the oscillations themselves are present as a consequence of spatial locking between states connected by such connections. Neither property is destroyed by small changes in $N$. These considerations appear relevant to the configuration studied by Tsitverblit et al. [38–40], who considered a liquid with a stable salt stratification in vertically elongated two-dimensional cells, heated from the side. In this problem the imposed temperature and concentration gradients are orthogonal. As a result the conduction state is absent and convection develops gradually as the Grashof number is raised. Despite this difference, Tsitverblit et al. identified a large variety of steady states whose complexity increased with increasing cell height. Among the states identified are a number of spatially localized states resembling those described here, raising the possibility that an explanation similar to that used here applies in the case of orthogonal imposed gradients as well, at least within the bistable regime.

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