Spatially localized states in natural doubly diffusive convection

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Numerical continuation is used to compute a multiplicity of stable spatially localized steady states in doubly diffusive convection in a vertical slot driven by imposed horizontal temperature and concentration gradients. The calculations focus on the so-called opposing case, in which the imposed horizontal thermal and solutal gradients are in balance. No-slip boundary conditions are used at the sides; periodic boundary conditions with large spatial period are used in the vertical direction. The results demonstrate the presence of homoclinic snaking in this system, and can be interpreted in terms of a pinning region in parameter space. The dynamics outside of this region are studied using direct numerical simulation. © 2008 American Institute of Physics.

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I. INTRODUCTION

Doubly diffusive convection, that is, convection driven by a combination of concentration and temperature gradients, is known to display a wealth of dynamical behavior whose properties depend both on the direction and the magnitude of the initial (or imposed) gradients.\textsuperscript{1,2} The earliest work was motivated by oceanographic applications\textsuperscript{3} and focused on situations in which the gradients are parallel to the buoyancy force.\textsuperscript{4} The latter is the situation of interest in the present paper. We measure the relative importance of the temperature and concentration gradients by the buoyancy ratio $N = \rho_L \Delta C / \rho_T \Delta T$, where $\rho_T < 0$, $\rho_C > 0$ are the thermal and solutal “expansion” coefficients, and $\Delta T$, $\Delta C$ are the differences in temperature and concentration imposed across the system. When the imposed gradients are parallel, the two buoyancy forces are said to be cooperating if $N > 0$ and opposing if $N < 0$. When $N = -1$, the opposing effects balance exactly and a pure conduction state is present. This is the case of interest in the present paper.

This paper is motivated by the observation by Ghorayeb and Mojtabi\textsuperscript{5} (see also Ref. 6) of an unusual single-cell state (labeled Type 1 in their Fig. 13) in a numerical study of the $N = -1$ case in a vertically extended container (aspect ratio 7). This (clockwise) cell is located at the center of the container, away from either end, and hence takes the form of what we call below a spatially localized state. To the best of our knowledge, the origin of this intriguing state has not hitherto been elucidated. We show here that the presence of this state is intimately related to the subcriticality of the primary instability of the conduction state (also demonstrated in Ref. 5) and that it is related to a phenomenon called homoclinic snaking.\textsuperscript{7} To do so, we examine the nature of the solutions in a vertical slot of width $\ell$ and height that is much larger than the width. We impose no-slip boundary conditions only along the sidewalls, and use periodic boundary conditions in the vertical direction. We choose this period, hereafter denoted $\Gamma \ell$, to be large in order to accommodate a substantial number of cells, but we follow Ref. 5 and restrict our attention to two-dimensional flows.

The system is described by the dimensionless equations

\begin{equation}
\frac{\partial \mathbf{u}}{\partial t} = - \nabla \cdot (\mathbf{u} \nabla) \mathbf{u} - \nabla p + \nabla^2 \mathbf{u} + \text{Gr}(T-C) \mathbf{e}_z, \tag{1}
\end{equation}

\begin{equation}
0 = \nabla \cdot \mathbf{u}, \tag{2}
\end{equation}

\begin{equation}
\frac{\partial T}{\partial t} = - \nabla \cdot (\mathbf{u} \nabla) T + \frac{1}{\text{Pr}} \nabla^2 T, \tag{3}
\end{equation}

\begin{equation}
\frac{\partial C}{\partial t} = - \nabla \cdot (\mathbf{u} \nabla) C + \frac{1}{\text{Sc}} \nabla^2 C, \tag{4}
\end{equation}

where $\mathbf{u} = (u, w, \mathbf{v})$ and $\nabla = (\partial_x, \partial_z)$ in $(x, z)$ coordinates, with $x$ in the horizontal direction and $z$ in the vertical direction. The Prandtl number $\text{Pr}$, the Schmidt number $\text{Sc}$, and the Grashof number $\text{Gr}$ are defined by

\begin{equation}
\text{Pr} = \nu / \kappa, \quad \text{Sc} = \nu / D, \quad \text{Gr} = \frac{g \rho_L \Delta T \ell^3}{\nu^2}, \tag{5}
\end{equation}

where $\nu$ is the kinematic viscosity, $\kappa$ is the thermal diffusivity, $D$ is the solute diffusivity, and $g$ is the gravitational acceleration. The boundary conditions read

\begin{equation}
\begin{align*}
\text{at } x = 0 &: \quad u = w = T = 1 = C = 1 = 0, \\
\text{at } x = 1 &: \quad u = w = T = C = 0,
\end{align*} \tag{6}
\end{equation}

together with periodic boundary conditions with period $\Gamma$ in the $z$ direction. The resulting equations are invariant under translations in $z$ and a $180^\circ$ rotation,

\begin{equation}
\Delta : (x, z) \to (1-x, -z), \quad (u, w, \Theta, \Sigma) \to -(u, w, \Theta, \Sigma), \tag{7}
\end{equation}

where we have written $T = 1 - x + \Theta$, $C = 1 - x + \Sigma$. It follows that the equations satisfied by $\Theta$ and $\Sigma$ possess $O(2)$ symmetry, the symmetry group of a circle under rotations (translations in $z$ mod $\Gamma \ell$) and reflection (rotation $\Delta$). Note that the
action of the reflection differs from that present when the layer is horizontal.

As a result, all steady-state bifurcations from the conduction state \( u = w = \Theta = \Sigma = 0 \) as \( \text{Gr} \) increases are pitchforks of revolution, and produce spatially periodic states with \( \Delta \) symmetry,

\[
(u, w, \Theta, \Sigma)(1 - x, -z) = - (u, w, \Theta, \Sigma)(x, z),
\]

cf. Ref. 8. In this paper, we use numerical branch following techniques to follow steady states with this symmetry from small amplitude near the primary instability threshold to large amplitudes. To do so, we use a numerical continuation method based on a Newton solver for the time-independent version of Eqs. (1)–(4) with the boundary conditions (6). The implementation of the method follows that of Tuckerman and Mamun and Tuckerman, but employs a spectral element method in which the domain \([0, 1] \times [0, \Gamma]\) is decomposed into \( N_e \) macro-elements of size \( [0, 1] \times [i\Gamma/N_e, (i+1)\Gamma/N_e] \), where \( N_e \) is the number of spectral elements and \( i \in \{0, \ldots, N_e-1\} \). In each element, the fields are approximated by a high-order interpolant through the Gauss–Lobatto–Legendre points. The Newton solver requires a first-order time integration scheme for the equations in conservation form; we use the scheme proposed by Karniadakis et al. in which the diffusive linear part of the equations is treated implicitly. Each time step, therefore, requires the inversion of four Helmholtz problems. This is carried out using a Schur factorization procedure on the weak form of the equations, a procedure that ensures the periodicity of the unknowns and their first derivative in the \( z \) direction.

Two types of states are of interest, namely the spatially periodic wave trains already mentioned and spatially localized states. In domains of finite spatial period \( \Gamma \), only the former bifurcate from the conduction state; the latter bifurcate in secondary pitchfork bifurcations from the periodic states, and do so already at small amplitude when \( \Gamma \) is large. Stability is examined with respect to \( \Delta \)-symmetric perturbations with period \( \Gamma \) only. All results are computed for \( \text{Pr} = 1 \), \( \text{Sc} = 1 \), and \( \text{Gr} = 10 \) or \( 15 \), depending on the branch and its symmetry properties.

II. RESULTS

Figure 1 shows the dimensionless kinetic energy as a function of \( \text{Gr} \) (details are shown in Figs. 2 and 3). The figure shows two primary branches of \( \Delta \)-symmetric spatially periodic states consisting of 12 (P1) and 9 (P2) pairs of rolls at onset. These bifurcate from the conduction state at \( \text{Gr} \approx 651 \) and \( \text{Gr} \approx 695 \), respectively. Other branches bifurcate from the conduction state between these two values (not shown), but none bifurcate prior to \( \text{Gr} \approx 651 \). Both branches bifurcate subcritically and turn around at larger amplitudes via saddle-node bifurcations (at \( \text{Gr} \approx 516.9 \) and \( \text{Gr} \approx 527.313 \), respectively). Figure 4 shows the evolution of the solutions along these branches as the amplitude (or kinetic energy) increases. The inclination of the cells at small amplitude is determined by linear theory, and hence directly
by the ratio $Sc/Pr$; as the cells become more vigorous the inclination is reduced, since the diffusive effects now compete with stronger convective transport. In addition, counterclockwise motions are rapidly suppressed in favor of enhanced clockwise circulation. In spite of this, both the solutions shown and their $\lambda/2$ translates have $\Delta$-symmetry with respect to the center of the domain.

Figure 1 shows two additional branches that bifurcate together from the first primary branch (P1) already at very small amplitude. These also consist of $\Delta$-symmetric states, but this time with period $\Gamma$ instead of $\lambda$. Initially the kinetic energies of these states are identical, but with decreasing Gr the energies “split” [Fig. 2(b)], and the two branches undergo a series of back and forth excursions that we refer to as *snaking*. Figures 2(a) and 3 show that both of these branches eventually terminate together at $Gr = 527.8$ on the second primary branch (P2), just above its saddle-node bifurcation at $Gr = 527.313$.

To understand the origin of the behavior shown in Fig. 1, we examine the nature of the solutions along the two snaking branches. Figure 5 (left panels) shows the solutions close to the initial bifurcation from the first primary branch (P1). We see that although the solutions retain the critical wavelength $\lambda \approx 12\lambda$, quite accurately (both consist of states with 12 pairs of rolls), their amplitude is now modulated in space, with period $\Gamma = 30$, in such a way that one solution (labeled L2) attains a maximum in the center of the domain, while the other (L1) attains a minimum in the center. Owing to the modulation, the resulting states are no longer related by translation by $\lambda/2$, nor are they related by translation by $\Gamma/2=15$: in L2, the maximum modulation amplitude coincides with the location of a maximum of the underlying periodic state, while in L1 the maximum of the modulation coincides with a minimum. Close to the initial bifurcation, the spatial modulation is slight, and so is the splitting in kinetic energy. With decreasing Gr, the modulation amplitude increases, resulting in states that resemble more and more spatially localized states, i.e., states in which convection occurs in only a portion of the domain, either in the center (L2) or near $x = \pm \Gamma/2$ (L1). This change produces a noticeable split in the energies, and when localization is complete, snaking sets in. Since the spatial resonance between the pattern wavelength $\lambda$ and the imposed period $\Gamma$ is weak, the secondary bifurcations at either end of L1, L2 will both be pitchforks.

The right panels in Fig. 5 show the solutions near their termination on the second primary branch. This branch corresponds to longer wavelength states, and the solutions are dominated by a clockwise circulation, with the counterclockwise rolls compressed into narrow shear zones, much as in Fig. 4(b), middle panel.

Figure 6 labels the saddle nodes proceeding up the L1 branch and descending down the L2 branch using integers $n$ to label the $n$th saddle node, hereafter SN$_n$. Figure 7 shows the flow at the saddle nodes on L1. At SN$_1$, the solution consists of a pair of clockwise rolls, separated by a quiescent state where convection is essentially absent, i.e., by SN$_2$ the spatial modulation has become so strong that convection is
completely spatially localized. As one passes from saddle node \( n \) to saddle node \( n+2 \), the solution adds a clockwise roll at either end, gradually filling the domain with clockwise circulating rolls. When the domain is almost full, the addition of further rolls is thwarted, and the branch executes a loop prior to terminating on the second primary branch of spatially periodic states [Fig. 2(a)]. The reason for the loop can be seen from Fig. 7: The "gap" remaining above \( \text{SN}_8 \) is not quite big enough to insert the ninth clockwise roll (let alone a pair of rolls) without compressing it. This in turn requires a certain wavelength adjustment among the rest of the rolls prior to the termination of the branch on the nine-roll branch. Figure 8 shows the corresponding evolution along \( L_2 \). The small loop is now replaced by the loop between \( \text{SN}_{10} \) and \( \text{SN}_9 \); this loop is larger since this time it is a pair of rolls that is being inserted prior to the termination of the branch. This loop is omitted from Fig. 8.

Figure 9 shows in greater detail the wavelength change visible already in Figs. 7 and 8 that takes place as one proceeds up the \( L_2 \) branch from \( \text{SN}_{15} \) to \( \text{SN}_{14} \) and then to \( \text{SN}_{13} \). We see that the wavelength is shorter at smaller \( \text{Gr} \) and longer at larger \( \text{Gr} \), and varies monotonically with \( \text{Gr} \) in between. A comparison with Fig. 4 reveals that these wavelengths differ in general from the wavelength of the coexisting spatially periodic solutions.

We now summarize the stability properties of the solutions described above. In all cases, we describe stability with respect to \( \Delta \)-symmetric perturbations with period \( \Gamma \) only. The
first primary branch of periodic states (P1) bifurcates sub-critically, and hence is initially once unstable. After the secondary bifurcation to the localized states, it becomes twice unstable; as the amplitude increases, there are four additional secondary bifurcations (not shown) before the saddle node is reached. Above the saddle node (Gr ≈ 516.9), the branch is once unstable and acquires stability by Gr ≈ 520.7. The branch remains stable until Gr ≈ 700. The behavior along the second primary branch computed (P2) is similar, except that this branch is initially 11 times unstable. With increasing amplitude, a succession of secondary bifurcations (not shown) reduces this number, so that the branch is only three times unstable below the saddle node (Gr ≈ 527.313), and twice above it. It then passes through another secondary bifurcation (Gr ≈ 527.5) after which it is once unstable. At Gr ≈ 527.8 the branch acquires stability, and a pair of (once unstable) branches of localized states bifurcate toward larger Gr (Fig. 3). We have not examined the secondary branches created in most of these bifurcations.

The localized states bifurcating from the first primary branch are initially once unstable, but both acquire stability at the first saddle node and then repeatedly lose and gain stability as the branches snake toward larger amplitude. As a result, both branches are just once unstable prior to their termination on the second primary branch at Gr ≈ 527.8, and the snaking interval (524 < Gr < 612) contains a multiplicity of stable localized states of different length, all coexisting with a pair of stable spatially periodic states, at least for Gr > 527.8. Instabilities with respect to bifurcations breaking the Δ symmetry (the so-called phase instabilities) will complicate this picture further. In particular, if these perturbations are admitted, the stability properties of L1 and L2 near their birth will be different, and likewise near their termination.15

III. DISCUSSION

The snaking process, whereby a localized state grows by nucleating additional cells on both sides while preserving the symmetry of the state, is familiar from simpler systems like the bistable Swift–Hohenberg equation, and its origin is quite well understood.15–17 Similar behavior was recently identified in binary fluid convection18 and is likely present in magnetoconvection as well.19

To understand the origin of snaking in the present system, we seek conditions under which time-independent solutions may exist that evolve with z from the conduction state \( u = w = \Theta = \Sigma = 0 \) at large negative z into a convecting state and then back into the conduction state as \( z \to \infty \). Whether such solutions are possible depends in part on the stability

![FIG. 8. Flow structure along the L2 branch from top to bottom. The panels show solutions near \( SN_n, n=11, \ldots, 18 \). Parameters are Pr=1 and Sc=11.](image)

![FIG. 9. Localized solutions between \( SN_{14} \) and \( SN_{13} \) on the L2 branch. The solutions between \( SN_{15} \) and \( SN_{14} \) are unstable while those between \( SN_{14} \) and \( SN_{13} \) are stable. Parameters are Pr=1 and Sc=11.](image)
properties of the conduction state in space: Eigenvalues \( q \) with positive real part indicate that a convecting state can grow from \( z = -\infty \), while eigenvalues with negative real part indicate that such a state may return, under appropriate conditions, back to the conduction state as \( z \to -\infty \). The \( \Delta \) symmetry of the equations is responsible for nongeneric properties of the spatial eigenvalues of this linear problem: Numerical solution of this problem near \( Gr = Gr_c \), shows that the four leading spatial eigenvalues form a quartet, \( \pm iq_c, \pm iO(\sqrt{r}) \), where \( r = Gr - Gr_c < 0 \) (Fig. 10), while for \( r > 0 \) they are all imaginary, \( \pm iq_c \). Here \( q_c = 2.5318 \) is the wavenumber of the primary instability selected at \( Gr = Gr_c = 650,9034 \). It follows that for \( r < 0 \), the conduction state is hyperbolic with two stable eigenvalues and two unstable eigenvalues. In contrast, for \( r > 0 \) all the eigenvalues lie on the imaginary axis and the conduction state is nonhyperbolic. In the latter case, exponentially localized states are not possible. At \( r = 0 \) there is a pair of imaginary eigenvalues \( \pm iq_c \), of double multiplicity, corresponding to a Hopf bifurcation in a spatially reversible system with 1:1 resonance. Analysis of the normal form of this bifurcation shows that there are precisely three solution branches that bifurcate from the conduction state at \( r = 0 \): Spatially periodic states and two types of spatially localized states. Figure 1 shows that all three bifurcate subcritically, and describes the development of the snaking region associated with the localized states as \( Gr \) decreases, and in particular the formation of the single-cell state originally identified in Ref. 5, cf. Fig. 8 near \( SN_{18} \). Within the theory, the width of the snaking region is controlled by a second parameter, the subcriticality of the periodic states: The snaking region originates in a codimension-2 point in this parameter space and is initially exponentially small, but broadens out as the subcriticality increases. In the theory, the snaking region is interpreted as the result of the pinning of the fronts at either end of the localized state to the periodic structure within. On the real line snaking continues forever, with each localized state adding rolls, a pair at a time, while preserving its \( \Delta \) symmetry. In the limit of infinitely large \( \Gamma \), the localized states high up the “snake” resemble a bound state of two fronts, the bottom one connecting the conduction state to the spatially periodic state and vice versa at the top. A state of this type corresponds to the simultaneous formation of a pair of heteroclinic connections, between the conduction state and the periodic state, and back to the conduction state. These connections are related by the \( \Delta \) symmetry, and correspond precisely to the bounding fronts. Such pairs of heteroclinic connections are present throughout the snaking or pinning region.

It should be noted that on a large but finite periodic domain, snaking cannot go on forever. When the localized state almost fills the domain, the branch of localized states turns over and terminates just above the saddle-node bifurcation on the branch of periodic states. In this region, the localized states resemble “holes” in an otherwise constant amplitude convecting state, a prediction confirmed in Figs. 5, 7, and 8, except for the fact that owing to the large wavelength at the upper boundary of the pinning region, the snaking branches terminate on a distinct branch of periodic states instead of terminating on the branch from which they initially bifurcate. In variational systems there is a simple intuitive picture of the wavelength selection process. Such systems possess a well-defined energy and evolve toward local energy minima. In these systems, the trivial (or zero) state has lower energy than the periodic state near the lower boundary of the pinning region, and the fronts bounding a localized state in this region are therefore forced inward against the pinning force arising from the periodic structure within. Such states are therefore compressed relative to the coexisting periodic state. In contrast, near the upper boundary of the pinning region, the energetically favored state is the periodic state, and the localized states expand. Much the same behavior is observed in the present problem, even though it is not variational.

It is of interest to examine the dynamics of the system at Grashof numbers just outside the pinning region. Figure 11 shows what happens just above the right boundary of this region, at \( Gr = 615.5 \). The solution is initialized with the steady state near \( SN_2 \) and allowed to evolve in time. Figure 11(a) shows the time evolution of both the kinetic energy integrated over the periodic domain and of the maximum of the horizontal velocity \( u_{\text{max}} \) within this domain. Both plots reveal a sequence of sharp “bursts,” characterized by an abrupt growth, followed by oscillatory relaxation to a new almost stationary state. Each burst is the manifestation of an explosive solute-driven nucleation of a pair of cells on either side of the existing localized structure followed by a marked slowing down in evolution as the system passes close to a saddle node. Once the system passes the saddle node, evolution speeds up and a new pair of cells is added in another burst. Figure 11(b) shows these bursts in the form of space-time plots, with Figs. 11(c) and 11(d) showing details of the first two bursts. The bursts are (initially) equally spaced in time because the saddle-node bifurcations that mark the boundary of the pinning region are aligned almost exactly vertically, and the process terminates only when the periodic domain is filled, i.e., a spatially periodic state is reached. Variational systems exhibit similar dynamics outside the pin-
generates a nonzero base state with a flow of the form \( \vec{v} \) according to whether Couette-like flow will add to or reduce the tilt of the cells. Gr\(=615.5 \) with Sc\(=11 \) and Pr\(=1 \).

Small departures from \( N=1 \) can be included as in Ref. 26; see also Ref. 27. In this case, the buoyancy term in Eq. (1) becomes Gr\((\Theta+N\Sigma)e_z+Gr(N+1)(1-x)\vec{e}_z \); the latter term generates a nonzero base state with a flow of the form \((0,W(x))\) whose strength increases with Gr. Two cases are of interest. In the first, we impose a zero flux requirement that forces a descending flow near \( z=0 \) to be balanced by a corresponding upflow near \( z=1 \) or vice versa. This additional Couette-like flow will add to or reduce the tilt of the cells according to whether \( N>-1 \) or \( N<-1 \). In this case, we can continue to impose \( \Delta \) symmetry on the solutions and hence can proceed as in the case \( N=-1 \). In particular, we expect the snapping behavior to remain robust. However, for large enough Gr and \( N \neq -1 \), the resulting shear flow can itself become unstable, and hence lead to new modes of instability; cf. Ref. 28. In this case, the stability properties of the localized states may differ from those computed here. In contrast, when the vertical flux is nonzero, all primary bifurcations [i.e., bifurcations from \((0,W(x))\)] turn into Hopf bifurcations \(29,30 \) and hence all periodic states in \( z \) will now travel along the slot. In this case we do not, in general, expect to see steady solutions, and the snapping behavior will be “unfolded.”

These considerations appear relevant to the configuration studied by Tsitverblit et al.,\(31–33 \) who considered a salt-water mixture with a stable salt stratification in vertically elongated two-dimensional cells, heated from the side. In this problem, the imposed temperature and concentration gradients are orthogonal. As a result, the conduction state is absent and convection develops gradually as the Grashof number is raised. Despite this difference, Tsitverblit et al. identified a large variety of steady states whose complexity increased with increasing cell height. Among the states identified are a number of spatially localized states resembling those described here, raising the possibility that a similar explanation to that used here applies in the case of orthogonal imposed gradients, at least within the bistable regime. Closely related phenomena are likely to be present in laterally heated vertical slots filled with a binary mixture with a negative Soret coefficient. This system, studied in Ref. 27 and 34, is closely related to the doubly diffusive system studied here.\(^{35} \) In both systems, the primary instability is subcritical, and both are spatially reversible in the sense described here. Thus the main ingredients for the formation of spatially localized structures are present here as well.

In this paper, we have computed a large variety of coexisting stable spatially localized states in natural doubly diffusive convection with opposing horizontal gradients of temperature and concentration, and provided an explanation for their presence in this system. States of this type were originally discovered by Gharyayeb and Mojtabi;\(^{36} \) related structures, now called convections, were subsequently identified in magnetococonvection\(^{36} \) as well as in binary fluid convection in both \(^{3} \)He\(^{4} \)He mixtures\(^{37} \) and water-ethanol mixtures.\(^{18} \)

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