Snakes and ladders: Localized states in the Swift–Hohenberg equation

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Abstract

The Swift–Hohenberg equation with cubic and quintic nonlinearities exhibits multiple stable and unstable spatially localized states of arbitrary length in the vicinity of the Maxwell point between spatially homogeneous and periodic states. The even and odd states are organized in a characteristic snaking structure and are connected by branches of mixed parity states forming a ladder-like structure. Numerical computations are used to illustrate the changes in the localized solutions as they grow in spatial extent and to determine the stability and wavelength of the resulting states.

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1. Introduction

There has been a great deal of interest recently in a phenomenon called homoclinic snaking. This term refers to the back and forth oscillations in a branch of steady spatially localized states sometimes referred to as dissipative solitons that are found in partial differential equations of sufficiently high order in space exhibiting bistability between a spatially homogeneous and a spatially periodic state. The best studied examples of such states occur in fourth order reversible partial differential equations [1–3]; often these equations are variational as well [4–6]. In many cases the presence of snaking can be traced to a degenerate spatial 1:1 reversible Hopf bifurcation from a spatially homogeneous state [1,2,7–9]; near this bifurcation snaking is a beyond-all-orders effect [10–14], but becomes visible further away from this bifurcation. The snaking occurs as the localized state adds ‘rolls’, symmetrically at both ends, and becomes more and more like a periodic wavetrain. In variational problems the snaking is related to the broadening of the Maxwell point between the homogeneous and periodic states.

This broadening is the result of pinning of the fronts that bound the localized state to the underlying periodic state [15].

Many of the above examples arise in problems which are naturally time-independent such as buckling problems [1,2,13]. However, the stability properties of localized states are of great importance for applications to pattern formation and fluid dynamics. In the present Letter we revisit the Swift–Hohenberg equation with cubic and quintic nonlinearities to provide a more detailed understanding of the simulations of this equation by Sakaguchi and Brand [16], and to discuss (a) the stability properties of the resulting solutions, and (b) the process of wavelength selection inside the localized states. As a result we identify a snaking structure that is more involved than hitherto assumed.

2. The Swift–Hohenberg equation

We study the Swift–Hohenberg equation in the form

\[ \partial_t u = ru - (\partial_x^2 + q_c^2)u + b_3 u^3 - b_5 u^5, \]  

where \( r \) is the control parameter and \( q_c, b_3 \) and \( b_5 \) are coefficients which we take to be \( q_c = 1, b_3 = 2, b_5 = 1 \). This equation possesses two important symmetries: it is invariant...
under \( x \to -x \), and \( u \to -u \). Moreover, in the presence of periodic boundary conditions with period \( L \), the equation possesses a Lyapunov functional \( F \) (which we refer to as an energy) given by

\[
F = \int_0^L dx \left\{ -\frac{1}{2} r u^2 + \frac{1}{2} [ (\partial_x^2 + q_c^2) u ]^2 - \frac{1}{4} b_3 u^4 + \frac{1}{6} b_5 u^6 \right\}
\]  

(2)

such that \( F_u = \frac{dF}{du} \). It follows that along any trajectory the energy decreases to a (local) minimum. In particular no Hopf bifurcations are possible and (at fixed \( x \)) all time-dependence ultimately dies out. These minima satisfy the time-independent version of (1). This equation is a fourth order dynamical system in \( x \) which, in appropriate coordinates, obeys Hamilton’s equations. The conserved (independent of \( x \)) Hamiltonian is

\[
H = -\frac{1}{2} (r - q_c^4) u^2 + \frac{1}{2} [ (\partial_x^2 + q_c^2) u ]^2 - \frac{1}{2} (\partial_x^2 u)^2 \\
+ \partial_x u \partial_x^2 u - \frac{1}{2} b_3 u^4 + \frac{1}{2} b_5 u^6.
\]  

(3)

The localized states of interest correspond to orbits of this system that are homoclinic to the trivial state \( u = 0 \). Such orbits must lie in the surface \( H = 0 \) [4,17].

The linear stability of a stationary solution \( u_s(x) \) of period \( L \) is determined by writing

\[
u(x, t) = u_s(x) + \epsilon \tilde{u}(x) e^{\beta t},
\]  

(4)

where \( \beta \) is the growth rate of the infinitesimal perturbation \( \epsilon \tilde{u}(x) \). Thus \( \tilde{u}(x) \) satisfies the eigenvalue problem

\[
\mathcal{L}[u_s(x)] \tilde{u}(x) = \beta \tilde{u}(x), \quad \tilde{u}(x + L) = \tilde{u}(x),
\]  

(5)

where

\[
\mathcal{L}[u_s(x)] \equiv \{ r - (\partial_x^2 + q_c^2)^2 \} + 3b_3 u_s^2(x) - 5b_5 u_s^4(x) \}.
\]  

(6)

Eq. (1) has up to five flat (spatially homogeneous) solutions, given by

\[
u_+ = \left( \frac{1}{2b_3} \left[ b_3 \pm \sqrt{b_3^2 + 4b_5 (r - q_c^4)} \right] \right)^{1/2},
\]  

\[
u_- = -\nu_+, \quad \nu_0 = 0,
\]  

(7)

and shown in Fig. 1. The trivial state \( \nu_0 \) loses stability with respect to spatially periodic perturbations with wavelength \( L_c = 2\pi/q_c \) at \( r_0 = 0 \) producing a branch \( \nu_P \) of spatially periodic states, and with respect to flat perturbations at \( r_2 \equiv q_c^4 \) producing \( \nu_{\pm} \). The latter are initially unstable but acquire stability with respect to flat perturbations at a saddle-node bifurcation at \( r_1 \equiv q_c^4 - b_3^2/4b_5 \), and to spatially periodic perturbations with wavelength \( L_c = 2\pi/q_c \) at \( r_\pm = \frac{1}{8} \left[ 10q_c^4 - b_3 \left( b_3 + \sqrt{b_3^2 + 4b_5 q_c^4} \right) \right] \) (see Fig. 1).

We compute the branches of spatially periodic solutions using numerical continuation [18]. The results, including stability, for the branch \( \nu_P \) created at \( r_0 \) are summarized in Fig. 2(a). There is always one zero eigenvalue corresponding to the eigenfunction \( \tilde{u} = u_s' \). All eigenvalues corresponding to eigenfunctions with wavelength \( L_c/m \) for integers \( m \geq 2 \) are negative, indicating that the patterned branch is stable with respect to short wavelength disturbances. Instabilities with respect to perturbations with wavelength \( m L_c, m \geq 2 \), occur only on the part of the branch between \( r_0 \) and the saddle-node at \( r_1 \) that is already unstable to perturbations with wavelength \( L_c \) (Fig. 2(a)). Fig. 2(b) shows that in the region of bistability there is one
Maxwell point: at \( r_{M1} \) the patterned state \( u_P \) has the same energy as the trivial flat state \( u_0 \). Other branches of \( L_c \)-periodic states, created at \( r_{\pm} \) on \( u_{\pm} \), extend all the way to \( r = \infty \) but are omitted.

Near the bifurcations at \( r_0 \) and \( r_\pm \) there exists a continuum of bifurcations to branches of solutions with wavelengths near \( L_c \). These branches look qualitatively similar to the wavelength \( L_c \) branch shown in Fig. 2(a). Perturbations with wavelength \( L_c \) are always the most dangerous instability for the flat states, and near such bifurcations the wavelength \( L_c \) branches are energetically favored over branches with other wavelengths. At large amplitude this is no longer the case, however, and on large domains we may expect the preferred solution to shift from this wavelength.

3. Localized states

The localized solutions of interest are small amplitude stationary states bi-symmetric to \( u_0 \), which exist near \( r = 0 \). The time-independent version of (1) forms a fourth order reversible dynamical system in space: the equation is invariant under the spatial reflection \( (x \rightarrow -x, u \rightarrow u) \). Any localized state \( u_{\ell}(x) \) connecting to the trivial state \( u_0 = 0 \) as \( x \rightarrow \pm \infty \) requires that \( u_0 \) has both stable and unstable spatial eigenvalues. It is easy to check that for \( r < 0 \) these eigenvalues are \( \pm \pm iq_c \pm (\sqrt{-r} - 2q_c) + O(r) \), while for \( r > 0 \) they are \( \pm \pm i(\sqrt{r} - 2q_c) + O(r) \). Thus for \( r < 0 \) the eigenvalues form a quartet, and \( u_0 \) is hyperbolic with two stable eigenvalues and two unstable eigenvalues. In contrast, for \( r > 0 \) all the eigenvalues lie on the imaginary axis and \( u_0 \) is not hyperbolic. As a result no exponentially localized states can be present when \( r > 0 \). At \( r = 0 \) there is a pair of imaginary eigenvalues \( \pm iq_c \) of double multiplicity. The bifurcation at \( r = 0 \) is thus a Hopf bifurcation in a reversible system with 1:1 resonance. Theory shows that under certain conditions the hyperbolic regime \( (r < 0, |r| \ll 1) \) contains a large variety of spatially localized states [8].

Four of these states can be constructed using perturbation theory. We define the small parameter \( \epsilon \) by \( r = -\epsilon^2 \mu_2, \mu_2 > 0 \), and look for stationary solutions of (1) of the form

\[
u(x) = \epsilon (A(X)e^{iq_c x} + c.c.) + O(\epsilon^2),\tag{8}
\]

where \( X \equiv \epsilon x \) is a large scale over which the amplitude of the pattern changes. It follows that

\[
4q_c^2 A_{XX} = \mu_2 A - 3b_3 A|A|^2 + O(\epsilon)\tag{9}
\]

Since \( b_3 > 0 \) the bifurcation at the origin is subcritical.

The simplest nontrivial solution of (9) is the uniform solution

\[
A(X) = \left(\frac{\mu_2}{3b_3}\right)^{1/2} e^{i\phi} + O(\epsilon),\tag{10}
\]

corresponding to spatially periodic states with period \( L_c \) near \( r = 0 \), viz.,

\[
u_P(x) = 2\left(\frac{-r}{3b_3}\right)^{1/2} \cos(q_c x + \phi) + O(r).\tag{11}
\]

Here \( \phi \) is an arbitrary phase and \( \mu_2 > 0 \) (so \( r < 0 \)). Other solutions to (9) can be found in terms of elliptic functions, and localized states correspond to infinite period solutions of this type with \( A \rightarrow 0 \) as \( X \rightarrow \pm \infty \):

\[
A(X) = \left(\frac{2\mu_2}{3b_3}\right)^{1/2} \text{sech} \left(\frac{X\sqrt{\mu_2}}{2q_c}\right) e^{i\phi} + O(\epsilon).\tag{12}
\]

This solution corresponds to

\[
u_{\ell}(x) = 2\left(\frac{-r}{3b_3}\right)^{1/2} \text{sech} \left(\frac{X\sqrt{-r}}{2q_c}\right) \cos(q_c x + \phi) + O(r).\tag{13}
\]

Like the spatially periodic states this family of solutions is parameterized by \( \phi \in S^1 \), which controls the phase of the pattern within the sech envelope. Within the asymptotics this phase remains arbitrary; there is no locking between the envelope and the underlying wavetrain at any finite order in \( \epsilon \). However, it is known [10,12,14,19] that this is no longer the case once terms beyond all orders are included. These terms break the rotational invariance of the envelope solution and result in a weak flow on the circle \( S^1 \). This flow in turn selects specific values of the phase: \( \phi = 0, \pi/2, \pi, 3\pi/2 \). At the same time these terms lead to transversal crossing of stable and unstable manifolds of \( u_0 \) [11,20] thereby producing the snaking that becomes so prominent farther away from \( r = 0 \). Note that the phases \( \phi = 0, \pi \) are the only two phases that preserve the symmetry \( (x \rightarrow -x, u \rightarrow u) \), while \( \phi = \pi/2, 3\pi/2 \) are the only two that preserve the symmetry \( (x \rightarrow -x, \bar{u} \rightarrow -\bar{u}) \). Since two of these have to be (weakly) attracting and the other two (weakly) repelling, it follows that four branches of localized states bifurcate subcritically from \( r = 0 \), two of which are stable with respect to translations of the envelope relative to the wavetrain, the others being unstable. However, all four states are amplitude-unstable [21]. The \( \phi = 0, \pi \) branches are related by the transformation \( u \rightarrow -u \) and hence have identical norms; likewise for the \( \phi = \pi/2, 3\pi/2 \) branches. Furthermore, near \( r = 0 \) all four branches are exponentially close but one might expect that with increasing \( -r \) the differences will grow and therefore that two branches with distinct norms will emerge from \( r = 0 \).

In the following we use the continuation package AUTO to extend the solutions (13) with \( \phi = 0, \pi/2, \pi, 3\pi/2 \) to solutions of (1) valid farther away from \( r = 0 \). Technically the numerical routine finds solutions to (1) on a large but finite domain \( L \), but provided the width of the resulting localized state is smaller than the domain (typically \( 40L_0 \)) the true homoclinic connection is well approximated by a large period orbit. Fig. 3 illustrates some of the results obtained in this manner for small \( -r \) while Fig. 4 extends these results to larger values of \( -r \). Sample profiles along each branch are shown in Fig. 5. Along the \( \phi = 0 \) branch the midpoint \( (x = 0) \) of the localized state is always a local maximum, while along the \( \phi = \pi \) branch the midpoint is always a local minimum. Along the \( \phi = \pi/2 \) branch the midpoint of the localized state always has maximum negative slope, while along the \( \phi = 3\pi/2 \) branch the midpoint always has maximum positive slope. Thus the \( \phi = 0, \pi \) solutions are even while the \( \phi = \pi/2, 3\pi/2 \) solutions are odd. Near the origin the ampli-
Fig. 3. (a) Bifurcation diagram showing the four branches of localized (homoclinic) states near $r = 0$. The branch of uniform patterned solutions $u_P$ is also shown. The lower panels show the homoclinic solutions on the (b) $\phi = 0$, (c) $\phi = \pi$, (d) $\phi = \pi/2$, and (e) $\phi = 3\pi/2$ branches at $r = -0.15$. The dotted lines in these panels show the leading order envelope computed in (13).

Fig. 4. (a) Bifurcation diagram showing the two homoclinic branches together with the flat and patterned branches of Fig. 2. Away from the origin the homoclinic branches are contained within the pinning region (shaded) between $r_{P1} \simeq -0.7126$ and $r_{P2} \simeq -0.6267$. Thick lines indicate stable solutions. The dashed vertical line marks the location of the Maxwell point $r_{M1}$ between the flat and patterned branches. (b) Closeup showing the ‘rungs’ connecting the snaking branches. Labels mark the locations of the profiles shown in Fig. 5.

The amplitude is small and the width of the sech envelope is large enough to contain many wavelengths of the underlying pattern. Away from the origin the amplitude grows and becomes comparable to the amplitude of the patterned states (specifically, the stable branch above the bifurcation at $r_3$) and the width decreases until it is comparable to $L_c$, the wavelength of the underlying pattern. Beyond this point both the even and odd branches undergo a series of saddle-node bifurcations responsible for the terminology homoclinic snaking. Each saddle-node bifurcation adds a pair of oscillations to the profile $u_\ell(x)$, and the saddle-node bifurcations asymptote exponentially rapidly to $r_{P1}$ and $r_{P2}$. At each value of $r$ within this range there exists an infinite number of solutions, each of a different width. Higher up along each ‘snake’ the solutions $u_\ell(x)$ begin to look like a pattern of wavelength $L_c$ and uniform amplitude, truncated at either end by a stationary ‘front’ of width of order $L_c$, connecting this state to $u_0$. The amplitude of this state is nearly identical to the upper branch of the patterned solutions. These results suggest that within the region $r_{P1} < r < r_{P2}$ there exist heteroclinic connections between the flat and patterned states as well. Far up each branch shown in Fig. 4, after many saddle-node bifurcations, the homoclinic solutions $u_\ell(x)$ connecting the flat state $u_0$ to itself resemble two of these heteroclinic connections, from $u_0$ up to the patterned state and then from the patterned state back down to $u_0$. These (Pomeau) fronts are stationary even away from $r_{M1}$ because of pinning by the underlying wavetrain [15]. Indeed we may think of the region $r_{P1} < r < r_{P2}$ as a Maxwell point that has been broadened by pinning of the underlying wavetrain [15]. Indeed we may think of the region $r_{P1} < r < r_{P2}$ as a Maxwell point that has been broadened by pinning of the underlying wavetrain [15]. Indeed we may think of the region $r_{P1} < r < r_{P2}$ as a Maxwell point that has been broadened by pinning of the underlying wavetrain [15]. Indeed we may think of the region $r_{P1} < r < r_{P2}$ as a Maxwell point that has been broadened by pinning of the underlying wavetrain [15]. Indeed we may think of the region $r_{P1} < r < r_{P2}$ as a Maxwell point that has been broadened by pinning of the underlying wavetrain [15].

Fig. 4 also indicates the stability of the localized solutions in time, a consideration that is absent from the general theory of reversible systems. The eigenvalue problem (5) yields the growth rate of infinitesimal perturbations of the homoclinic solutions at each point along the branches, as well as the associated eigenfunctions $\tilde{u}(x)$. The eigenfunctions that play a critical role in what follows are localized around the base state $u_\ell(x)$ and are therefore insensitive to the exact choice for $L$. The results for the even ($\phi = 0, \pi$) and odd ($\phi = \pi/2, 3\pi/2$) branches are shown in Fig. 6. Near the origin the even branches are unstable to even perturbations, while the odd branches are unstable to both odd and even perturbations. Each zero crossing
of the mode with the same parity as the profile $u_\ell(x)$ (the amplitude eigenvalue) corresponds to a saddle-node bifurcation in Fig. 4, while each zero crossing of the opposite parity mode (the phase eigenvalue) indicates a pitchfork bifurcation to a pair of branches of asymmetric states. On all branches the phase eigenvalues remain almost zero until the snakes develop. Once this happens the stability is controlled by a single even mode and a single odd mode—several other modes approach zero growth rate but never cross. In this regime the phase eigenvalues are responsible for the presence of short branches of asymmetric states that connect the even and odd states. Higher up the snake Fig. 6 shows that the bifurcations to these states occur closer and closer to the saddle-node bifurcations, but always on the unstable side. Thus the asymmetric states are always once unstable. Thus the complete structure of the snaking region consists of a (double) pair of snaking branches connected by approximately horizontal ‘rungs’ forming a ladder-like structure (Fig. 4).

Fig. 7 shows the three types of dangerous eigenfunctions. For extended localized states the amplitude and phase eigenfunctions are localized near the Pomeau fronts; each state is in addition neutrally stable with respect to the Goldstone mode $u'_\ell$.

Although the snaking of the true homoclinic orbit represented in Fig. 4 goes on forever, the solutions shown in the figure were found on a finite domain in $x$. In these circumstances the sequence of saddle-node bifurcations must terminate: as the domain fills with the pattern the snakes terminate on one of the patterned branches, determined in part by the domain length. The termination point corresponds to a bifurcation at which a pair of homoclinic branches biasymptotic to the patterned state is created, similar to the bifurcation at the other end of the snake at $r = 0$. This duality is a generic feature in problems of this type [22]. Numerical analysis beyond that presented here indicates that in the limit $L \to \infty$ the location of this bifurcation coincides with the saddle-node bifurcation at $r_3$ [23].

4. Wavelength selection

Close examination of the wide homoclinic solutions (far up the snakes) shows that despite appearances the shape inside the envelope does not match the patterned branch perfectly. The largest deviation is in the wavelength. Recall that on a large domain the wavelength of the energetically preferred pattern will deviate from $L_c$. Fig. 8(a) shows the wavelength of the preferred spatially periodic solution in the neighborhood of the pinning region, obtained by minimizing the energy density with respect to the spatial period $L$, at fixed $r$. The deviation from $L_c$ is small, typically less than one part in $10^3$. Fig. 8(a) also shows the wavelength of the pattern within the localized states, whose
deviation from $L_c$ is typically much larger. Far up the snaking branch, where the localized state contains many wavelengths of the pattern, this wavelength is spatially uniform and independent of the width of the localized state. It is not independent of $r$, however. Pomeau’s pinning mechanism allows for the existence of localized states away from the Maxwell point, but evidently the wavelength of the patterned domain within such states is affected by the presence of the fronts at either side: near $r_{P1}$, where the flat state is energetically favored, the packet is squeezed tighter, while near $r_{P2}$, where the patterned state is favored, the packet expands [16]. This is a frustration effect: the fronts at $r_{P1}<r<r_{M1}$ want to move in such a way as to eliminate the localized state, leading to a compression of the state relative to its wavelength at the Maxwell point $r=r_{M1}$ (Fig. 8). Likewise at $r_{M1}<r<r_{P2}$ the fronts want to move outwards, thereby stretching the localized state. It is noteworthy that the resulting compression or expansion is distributed uniformly across the localized state, a fact that appears to be a consequence of local energy minimization. As a consequence the localized states approach a different spatially periodic state at each $r$, although in Fig. 4 we use the wavelength $L_c$ patterned branch as a stand-in for the actual periodic state approached along the snake.

The wavelength selection mechanism just described is a consequence of the requirement that the localized states correspond to homoclinic orbits in the surface $H=0$. Fig. 8(b) shows that the pattern within the envelope does indeed satisfy $H=0$. In contrast, the Hamiltonian evaluated along the patterned branch selected by energy minimization is in general nonzero (Fig. 8(b)). It follows that the wavelength within the localized states must shift to accommodate the requirement $H=0$, and it is this requirement that predicts the wavelength change anticipated by Pomeau [4,17,24]. In particular, the twin conditions $F=H=0$ determine the location $r_{M1}$ of the Maxwell point and the associated wavelength; Fig. 8(a) shows that this wavelength is almost $L_c$ justifying a posteriori the use of this wavelength in Fig. 2.

5. Localized states as a function of $b_3$

Thus far we have studied the behavior of (1) as $r$ varies at fixed values of $q_c$, $b_3$, and $b_5$. To explore the $b_3$ dependence of our results we rescale $x$ and $u$ so that $q_c=1$ and $b_5=1$, and vary $b_3>0$. Fig. 9 summarizes the region of existence of heteroclinic connections to the trivial flat state $u_0$. Near the codimension-two bifurcation $(0,0)$ the pinning region is very narrow and the Maxwell point is well approximated by the heteroclinic orbit from normal form theory [7]. At larger values of $b_3$ the pinning region widens, and the normal form results no longer apply. A better approximation can be made by taking advantage of the fact that the Swift–Hohenberg equation is variational, and the fact that the patterned solution $u_P$ are dominated by a single Fourier mode over the parameter range of interest here:

$$u_P(x) \approx a \cos(q_c x),$$  

where the amplitude $a$ varies with $r$. The energy of the function (14) is

$$F(a) = -\frac{r}{4} a^2 - \frac{3b_3}{32} a^4 + \frac{5b_5}{96} a^6.$$  

The extrema of the energy are found at

$$a_{\pm} = \left( \frac{3b_3 \pm \sqrt{9b_3^2 + 40b_5 r}}{5b_5} \right)^{1/2},$$

corresponding to the stable ($a_+$) and unstable ($a_-$) patterned branches of Fig. 2. The energy of the $u_0$ flat branch is zero by definition, so the Maxwell point $r_{M1}$ between this and the $a_+$ patterned branch satisfies $F(a_+)_{r=r_{M1}} = 0$. Thus $r_{M1} = -\frac{27b_3^2}{1600b_5}$.
Because this result is derived from global considerations it is valid farther from the codimension-two point. In fact, it is nearly indistinguishable from the exact Maxwell point over the entire range of parameters plotted in Fig. 9.

The bifurcation diagram in Fig. 4 corresponds to a horizontal slice through this figure at \( b_3 = 2 \) and is typical of the behavior below \( b_3 \gtrsim 3.521 \). Above this value of \( b_3 \), a new Maxwell point, corresponding to equal energies of the \( u_0 \) and \( u_+ \) states,

\[
r_{M2} = q_c^4 - \frac{3b_3^2}{16b_5},
\]

becomes dynamically important. For \( b_3 \gtrsim 3.521 \) the new Maxwell point enters the pinning region around \( r_{M1} \) and the structure of the flat and patterned states changes, as do the homoclinic branches. In particular, the four homoclinic branches created at the origin undergo homoclinic snaking towards the \( u_+ \) state instead of \( u_0 \). Since \( u_+ \) is a spatially homogeneous state no pinning occurs, and the snakes collapse asymptotically to a single point at \( r = r_{M2} \). Thus at \( r_{M2} \) an infinite number of homoclinic states of different lengths biasymptotic to \( u_0 \) is still present, but away from \( r_{M2} \) only a finite number of such states remains [25]. Bifurcation diagrams describing these homoclinic states are shown in Fig. 10.

6. Conclusions

In this Letter we have revisited the properties of spatially localized states within the Swift–Hohenberg equation with cubic and quintic nonlinearities discovered by Sakaguchi and Brand [16]. This system is reversible, variational, and exhibits bistability between the trivial state \( u_0 = 0 \) and spatially periodic patterns when \( b_3 > 0 \). We have examined the behavior of the four branches of spatially localized states emanating from the codimension-two point \((r, b_3) = (0, 0)\), and tracked the evolution of the solutions along these branches. With increasing \(|r|\) we observed characteristic snaking associated with the broadening of the Maxwell point due to pinning of the fronts bounding the localized states to the underlying periodic structure. A careful study of the stability properties of these solutions revealed the importance not only of amplitude perturbations that are responsible for the sequence of saddle-node bifurcations associated with the snakes, but also of parity breaking bifurcations that are responsible for branches of asymmetric states that connect the branches of odd and even states that make up the snake. These connections resemble the rungs of a ladder and are responsible for the snakes-and-ladders structure of the pinning region. Related structures have been seen before in the context of double-pulse states [13,26] but not for the single-pulse states considered here. In addition we saw that the tendency of the fronts at either end of the localized state to move when \( r \) is away from the Maxwell point leads to a significant compression \((r < r_{M1})\) or dilation \((r > r_{M1})\) of the wavelength of the localized state, and demonstrated that this wavelength change is computed correctly from the requirement that \( H = 0 \). Related behavior occurs in the quadratic/cubic Swift–Hohenberg equation as well [7].

Localized states also occur in nonvariational problems, both in one [27–34], two [11, 31, 35–40] and three [41] dimensions. In many of these cases snaking has been observed. However, whether the interconnecting branches of asymmetric states are present in these systems remains to be seen, cf. [31,42].

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