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Parity-breaking bifurcation in inhomogeneous systems

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Received 19 August 1996, in final form 12 June 1997
Recommended by H Levine

Abstract. The steady-state reflection-breaking bifurcation from a circle of nontrivial equilibria in O(2)-equivariant systems results in a pair of travelling waves. When the continuous part of the group O(2) is weakly broken, the corresponding instability may lead to nonsymmetric but steady states. The transition from this state to the travelling wave state with increasing bifurcation parameter is complex, and typically involves sequences of global bifurcations. The possible scenarios are described in detail and the results are related to the dynamics associated with parity-breaking instabilities of spatially periodic patterns in inhomogeneous systems.

PACS numbers: 4754, 4720, 0220

1. Introduction

The presence of symmetry is responsible for much of the novel dynamical behaviour associated with equivariant systems. Such systems typically exhibit spontaneous symmetry-breaking bifurcations; when this happens the resulting solutions have less than full symmetry, and hence occur in group orbits. Group orbits of solutions are of particular interest when the symmetry that is broken is continuous. In this case the solutions are no longer isolated, and this fact has a strong influence on any subsequent bifurcations. In particular, steady-state bifurcations from a group orbit of solutions may result in dynamical behaviour. Perhaps the simplest example of this phenomenon arises when the symmetry group is O(2). The presence of this group forces the existence of a trivial, O(2)-symmetric state. As a parameter is varied this state may lose stability at a symmetry-breaking steady-state bifurcation, producing a circle of $Z_2$-symmetric states. In this case a subsequent steady-state reflection-breaking bifurcation produces a flow along the group orbit, and hence leads to a travelling wave (TW). This bifurcation has found a number of applications [28, 36, 10, 33, 27, 38, 6, 13] and is of theoretical interest precisely because it provides an easily realized example of a steady-state bifurcation producing time dependence. In the physics literature this bifurcation is usually referred to as a parity-breaking or infinite period bifurcation [5, 12, 34].

In many cases, however, the O(2) symmetry is present as a result of an idealization of the physical system and is likely to be influenced by imperfections. Imperfections which do not break the symmetry merely change the coefficients of the basic normal form slightly and therefore have no effect on the generic problem. Imperfections that break only discrete symmetries are more interesting but do not lead to an increase in the dimensionality of
the problem; consequently their effects are less dramatic. Of greatest interest are effects arising from the breaking of continuous symmetries. These lead to a coupling between the spatial phase and the dynamics ‘orthogonal’ to it. The resulting dynamical behaviour is therefore much richer. Consequently, in this paper we focus on the effects of small imperfections on the parity-breaking bifurcation that break the continuous part of O(2) while preserving the reflection symmetry \( Z_2 \). One expects that close to the bifurcation from the group orbit of steady states the imperfections will dominate and hence lead to secondary bifurcations to nonsymmetric but steady states. This phenomenon may be described as ‘pinning’ of the pattern. On the other hand, farther from the bifurcation the influence of the imperfections should decrease, and the result, based on the presence of O(2) symmetry, should be recovered, i.e. one expects that the pattern will unpin and begin to drift. As in other problems of this type (Dangelmayr and Knobloch [9], Hirschberg and Knobloch [17]) one finds that the transition between these two types of behaviour can be remarkably complex, and typically involves global bifurcations. This paper is devoted to describing the manner in which this transition is accomplished. Because the phase space of the problem is a cylinder, new types of transitions are allowed that cannot otherwise take place.

Because of the continuous SO(2)-symmetry of the original system equations (1) are invariant under translations \( \phi \rightarrow \phi + \phi_0 \), any \( \phi_0 \), and the phase \( \phi \) decouples from the \( c \)-equation. The invariance under \( (c, \phi) \rightarrow (c, -\phi) \) is a consequence of the reflection symmetry. When the translation invariance is broken, the phase no longer decouples and additional terms must be included that lead to a coupling to the phase. The form of these coupling terms must respect reflection invariance, resulting in a system of the form

\[
\dot{\phi} = c + \varepsilon f_1(\phi) + O(|\mu c| + |c|^3)
\]

\[
\dot{c} = (\mu - c^2)c + O(|c|^5 + |\mu^2 c|)
\]

where the bifurcation parameter \( \mu \) is chosen such that the bifurcation point is at \( \mu = 0 \), cf Dangelmayr and Knobloch [8], Greene and Kim [14], Bensimon et al [3], Knobloch and Moore [22]. The first equation describes the drift along the group orbit. Consequently, \( \phi (0 \leq \phi < 2\pi) \) can be identified with the (spatial) phase of the pattern, and \( c \) with its phase velocity. The second equation shows that the bifurcation to TW is a pitchfork, and leads to either clockwise \( (c < 0) \) or anticlockwise \( (c > 0) \) drift. As a consequence of the continuous SO(2)-symmetry of the original system equations (1) are invariant under translations \( \phi \rightarrow \phi + \phi_0 \), any \( \phi_0 \), and the phase \( \phi \) decouples from the \( c \)-equation. The invariance under \( (c, \phi) \rightarrow (c, -\phi) \) is a consequence of the reflection symmetry.

When the translation invariance is broken, the phase no longer decouples and additional terms must be included that lead to a coupling to the phase. The form of these coupling terms must respect reflection invariance, resulting in a system of the form

\[
\dot{\phi} = c + \varepsilon f_1(\phi) + O(|\mu c| + |c|^3 + \varepsilon^2 + |\varepsilon c|)
\]

\[
\dot{c} = (\mu - c^2)c + \varepsilon f_2(\phi) + \varepsilon f_3(\phi) + O(|c|^5 + |\mu^2 c| + \varepsilon^2 + |\varepsilon c^2|)
\]

where \( \varepsilon \ll 0 \) measures the strength of the symmetry-breaking effects, and \( f_1, f_3 \) are odd \( 2\pi \)-periodic functions, while \( f_2 \) is even. In section 2 we derive these equations using invariant manifold theory and then focus our attention on a particular choice of the functions \( f_1, f_2, f_3 \), approximating them by the first term in their respective Fourier series. Specifically, omitting the error terms, we analyse the following dynamical system posed on the cylindrical phase space \( \mathbb{R} \times S^1 \),

\[
\dot{\phi} = c - \varepsilon \alpha \sin \phi
\]

\[
\dot{c} = (\mu - c^2)c + \varepsilon \beta c \cos \phi + \varepsilon \gamma \sin \phi
\]

where \( \alpha, \beta, \gamma \in \mathbb{R} \). It should be noted that other choices for \( f_1, f_2, f_3 \) may also be of interest. For example, choosing the period of the imperfection to be a fraction of \( 2\pi \) leads to a picture of the dynamics much like that described in this paper, but with more fixed points selected from the circle of equilibria.
In section 3 we summarize the general properties of equations (3), regarding this system as an unfolding of (1). In this analysis we do not need to specify the relative orders of magnitude of the parameters \(\alpha, \beta, \gamma\) and \(\mu\). Some aspects of (3) are described by Knobloch et al [23]; this paper focuses on the subsidiary bifurcations that occur for special values of the parameters and on their local unfoldings. These points provide a key to the global picture of the transition between the symmetric and nonsymmetric steady states and the drifting ones, and allow us to study its parameter dependence; this analysis forms the bulk of sections 5 and 6. In section 7 we summarize our results and discuss their experimental implications.

2. Perturbed invariant manifold for the parity-breaking instability

In this section we derive equations (2) from first principles. The techniques we use, invariant manifold theory, make explicit the construction used by Krupa [24] to study bifurcations from group orbits of equilibria, and provide a systematic approach to problems of this type. Readers not interested in the details of the derivation may skip this section; the analysis of equations (3) that follows in sections 3–6 is independent of the present section.

Since the parity-breaking instability involves a bifurcation from an entire circle of equilibria standard centre manifold reduction does not apply. Instead, the flow is restricted to an invariant manifold whose approximate form can be calculated perturbatively. We start from a system of the form

\[
\dot{u} = F(u, \mu) + \varepsilon G(u) \tag{4}
\]

for \(u \in \mathcal{H}\), a suitable real Hilbert space with an inner product \((\cdot, \cdot)\). We consider an orthogonal action of O(2) in \(\mathcal{H}\), with the SO(2)-part represented by a family of linear operators \(\{T_\phi : -\pi \leq \phi \leq \pi\}\), and the \(Z_2\)-part is determined by a reflection operator \(R : \mathcal{H} \to \mathcal{H}\), such that \(R^2 = \text{id}\), the identity in \(\mathcal{H}\). Orthogonality of the O(2)-action implies that these operators satisfy the relations

\[
T_\phi^* = T_{-\phi}^{-1} \quad R^* = R = R^{-1} \quad T_\phi R = RT_{-\phi} \tag{5}
\]

where the asterix refers to the corresponding adjoint operator. The basic hypothesis is that \(F\) in (4) is equivariant under O(2), while \(G\) is only equivariant under the \(Z_2\)-part of O(2),

\[
F(T_\phi u, \mu) = T_\phi F(u, \mu) \quad F(Ru, \mu) = RF(u, \mu) \quad G(Ru) = RG(u). \tag{6}
\]

The term \(\varepsilon G(u)\) in (4) is considered to be a small symmetry-breaking imperfection, with \(|\varepsilon| \ll 1\) measuring the magnitude of the symmetry-breaking effects. We refer to (4) with \(\varepsilon = 0\) and \(\mu = 0\) as the perfect system (full O(2)-symmetry) and the imperfect system (broken SO(2)-symmetry), respectively. The perfect system may, for example, describe an azimuthally symmetric PDE posed on a circular disk or annulus, with \(\varepsilon G(u)\) accounting for spatial inhomogeneities or slight deformations of the spatial domain.

Next we have to specify the conditions for a parity-breaking instability at \(\mu = 0\) in the perfect system. We assume that, for \(\varepsilon = 0\) and \(\mu = 0\), (4) possesses a natural, reflection-invariant steady state by virtue of reflection invariance, i.e.

\[
F(u_0, 0) = 0 \quad Ru_0 = u_0 \quad \text{and} \quad \frac{d}{d\phi} T_\phi u_0|_{\phi=0} = Su_0 \neq 0. \tag{7}
\]

In the third equation of (7) we have introduced the generator of SO(2), \(S\), which satisfies

\[
T_\phi = e^{i\phi S} \quad SR + RS = 0 \quad (S^2)^* = S^2 \tag{8}
\]
with eigenvalues \( i_n, n \in \mathbb{Z} \). We set
\[
d = \| Su_0 \| \quad w_0 = \frac{1}{d} Su_0
\] (9)
and note that \( w_0 \) is odd under \( R, Rw_0 = -w_0 \), since by assumption \( u_0 \) is even.

Let
\[
L = d_u F(u_0, 0)
\] (10)
be the linearized operator at \( u_0 \). This operator has a nontrivial nullspace because \( Lw_0 = 0 \) by virtue of the SO(2)-symmetry. A parity-breaking bifurcation takes place if the generalized eigenspace corresponding to the zero eigenvalue of \( L \) is two-dimensional and \( L \) is represented by a nilpotent \( 2 \times 2 \) Jordan block in this space (e.g. Aston et al [1]). In this case there is a \( w_1 \in \mathcal{H} \) with \( Lw_1 = w_0 \), and we may choose \( w_1 \) such that \( (w_1, w_0) = 0 \).

Similarly, the adjoint \( L^* \) has a two-dimensional generalized eigenspace corresponding to the zero eigenvalue. Here we only need the spanning element for the null space, denoted by \( w^*_1 (L^* w^*_1 = 0) \), which is orthogonal to \( w_0 \), \( (w^*_1, w_0) = 0 \); this element may be normalized such that \( (w^*_1, w_1) = 1 \).

Let \( \mathcal{H}_0 \) be the orthogonal complement of \( w_0 \) and let \( Q_0 \) be the orthogonal projection onto \( \mathcal{H}_0 \),
\[
\mathcal{H}_0 = \{ v \in \mathcal{H} | (v, w_0) = 0 \} \quad Q_0 = \text{id} - (\cdot, w_0)w_0.
\] (11)
We introduce polar-type coordinates \( u \rightarrow (\phi, v) \) for \( u \) by setting
\[
u = T_\phi(u_0 + v) \quad v \in \mathcal{H}_0.
\] (12)
This equation provides a change of coordinates in a tubular neighbourhood of the group orbit of equilibria, i.e. \( (\phi, v) \in [-\pi, \pi) \times B \), where \( B \) is a ball in \( \mathcal{H}_0 \) with a sufficiently small radius. Substituting (12) into (4) yields
\[
\dot{v} + \phi (d w_0 + Sv) = F(u_0 + v, \mu) + \varepsilon T_{-\phi} G(T_\phi(u_0 + v)).
\] (13)
Operating on (13) by \( (\cdot, w_0) \) and by \( Q_0 \) leads to the following evolution system for the new coordinates \( (\phi, v) \),
\[
\dot{\phi} = F_1(v, \mu) + \varepsilon G_1(\phi, v) \\
\dot{v} = F_2(v, \mu) + \varepsilon G_2(\phi, v)
\] (14)
where
\[
F_1(v, \mu) = \frac{1}{D} (F(u_0 + v, \mu), w_0) \\
G_1(\phi, v) = \frac{1}{D} (G(T_\phi(u_0 + v)), T_\phi w_0) \\
F_2(v, \mu) = Q_0 F(u_0 + v, \mu) - F_1(v, \mu)Q_0 Sv \\
G_2(\phi, v) = Q_0 T_{-\phi} G(T_\phi(u_0 + v)) - G_1(\phi, v)Q_0 Sv
\]
and
\[
D = d + (Sv, w_0).
\]
We note that (14) is equivariant under \( (\phi, v) \rightarrow (-\phi, Rv) \). Moreover, the phase decouples from the equation for \( v \) when \( \varepsilon = 0 \). The assumptions on \( L \) imply that the linearization
\[
L_0 = d_v F_2(0, 0) = Q_0 L|_{\mathcal{H}_0}
\] (15)
of the \( v \)-system has a simple zero-eigenvalue with eigenvector \( w_1 \) and co-eigenvector \( w^*_1 \). Hence this system can, for \( \varepsilon = 0 \), be reduced via centre manifold reduction in the standard
way (see, e.g. Carr [4]). To this end one introduces a (in general nonorthogonal) projection $Q$ in $\mathcal{H}_0$,

$$Q = \text{id} - (\cdot, w_1^*)w_1 \quad \mathcal{H}_0' = Q \mathcal{H}_0,$$

(16)
splits $v$ into $v = cw_1 + w$ with $w \in \mathcal{H}_0'$, and represents the centre manifold as a graph,

$$w = W(c, \mu) = w_{20}c^2 + O(c^3).$$

(17)
The Taylor expansion of $W$ is uniquely determined by the invariance condition $\dot{w} = e^{3iW}/\epsilon$ and can be calculated to any desired order in a perturbation calculation. With the following Taylor expansion

$$F(u_0 + v, \mu) = \mu F_{01} + (L + \mu F_{11})v + \frac{1}{2} F_2[v, v] + \frac{1}{6} F_3[v, v, v] + \text{h.o.t.}$$

(18)
one obtains to third order

$$\dot{c} = \mu a_{11}c + a_{30}c^3 + O(|\mu c^3| + |c^5| + |\mu^2 c|)$$

(19)
where

$$a_{11} = (F_{11}w_1, w_1^*)$$

$$a_{30} = \frac{1}{2}(F_2[w_1, w_{20}], w_1^*) + \frac{1}{6}(F_3[w_1, w_1, w_1], w_1^*)$$

$$- \frac{1}{d}\left(Lw_{20} + \frac{1}{2} F_2[w_1, w_1] - \frac{1}{d} Sw_1, w_0 \right)(Sw_1, w_1^*).$$

The coefficient $w_{20}$ is determined by

$$L_0' w_{20} + \frac{1}{2} Q F_2[w_1, w_1] - \frac{1}{d} QS w_1 = 0$$

where $L_0' = Q L |_{\mathcal{H}_0'}$ is invertible in the space $\mathcal{H}_0'$. The dynamics of the phase (for $\epsilon = 0$) then follows by substituting $v = cw_1 + W(c, \mu)$ into $F_1$ and leads directly to (1). Assuming that $a_{30} < 0$ and $a_{11} \neq 0$, a suitable rescaling of $t, \mu, c$ leads finally to the unperturbed normal form (1).

By rotating the local centre manifold determined at $u_0$ along the group orbit of equilibria a cylindrical invariant manifold is swept out. Being normally hyperbolic, this manifold also persists for sufficiently small $\epsilon \neq 0$ and is amenable to an expansion in $\epsilon$. The calculation proceeds as before, the difference being only that now variations in $\phi$ also have to be taken into account. The perturbation ansatz for the invariant manifold reads

$$w = W(c, \mu) + \epsilon \tilde{W}(c, \phi, \mu) + O(\epsilon^2) \in \mathcal{H}_0'$$

(20)
where $W$ is the centre manifold function derived above and $\tilde{W}$ may be expanded in a Taylor series with respect to $(c, \mu)$, but now with coefficients that are $2\pi$-periodic $\mathcal{H}_0'$-valued functions. These functions may also be successively determined from the invariance requirement. For our purposes $W$ and $\tilde{W}$ are needed up to order

$$w = w_{20}c^2 + \epsilon w_{00}(\phi) + O(|c^3| + |\mu| + |\epsilon||(c| + |\mu|) + \epsilon^2)$$

(21)
which fully determines the expansion of the perturbed normal form as stated in equation (2). Substituting this ansatz into the invariance condition $\dot{w} = e^{3iW}/\epsilon$ leads to an equation for $\tilde{w}_{00}$,

$$L_0' \tilde{w}_{00} + QT_{-\phi} G(T_\phi u_0) = 0.$$
where

\[ P(\phi) = (L\tilde{w}_{00}, w_0) + (T_{-\phi} G(T_{\phi} u_0), w_0) \]
\[ D(\phi) = (Q_0 T_{-\phi} G(T_{\phi} u_0), w_0^*) \]
\[ E(\phi) = \frac{1}{2} (F_2[w_1, \tilde{w}_{00}] + F_2[\tilde{w}_{00}, w_1], w_0^*) - \frac{1}{d} P(\phi)(Sw_1, w_0^*) + \frac{1}{d} (S\tilde{w}_{00}, w_0^*) + (T_{-\phi} d_u G(T_{\phi} u_0) T_{\phi} w_1, w_0^*) . \]

A suitable rescaling of \( t, \mu, c \) leads to equations (2).

3. General properties of the perturbed equations

We first summarize some basic properties of equations (3). For this analysis we set \( \varepsilon \alpha = \epsilon, \varepsilon \beta = \delta, \varepsilon \gamma = \eta \), and treat \( (\epsilon, \delta, \eta, \mu) \in \mathbb{R}^4 \) as genuine small parameters which may vary in a full neighbourhood of the origin, as in Knobloch et al [23]. The symmetry-breaking terms select two steady states from the circle of steady states, \( (c, \phi) = (0, 0), (0, \pi) \), hereafter referred to as SS\(_{0,\pi}\). These reflection-symmetric solutions are always present and take the place of the circle of solutions \( (c, \phi) \in [0, 2\pi) \), in the translation-invariant system. The stability of these solutions is described by the characteristic equation

\[ s^2 - (\mu \mp \epsilon \pm \delta)s \mp \epsilon (\mu \pm \delta) \mp \eta = 0 \]  

respectively, so that SS\(_{0,\pi}\) undergo steady-state bifurcation when

\[ \mu = \mp \delta - \frac{\eta}{\epsilon} \]  

and a Hopf bifurcation when

\[ \mu = \pm (\epsilon - \delta) \]  

provided only that \( \epsilon^2 + \eta < 0 \). The steady-state bifurcations are pitchfork bifurcations and produce pairs of asymmetric equilibria; the Hopf bifurcations produce symmetric limit cycles.

The nontrivial (i.e. asymmetric) fixed points \( P^\pm \) take the form \( (c, \phi) = (c_0, \phi_0) \), and are given by

\[ \mu + \delta \cos \phi_0 - c_0^2 + \frac{\eta}{\epsilon} = 0 \]
\[ c_0 = \epsilon \sin \phi_0 . \]  

These undergo a saddle-node bifurcation when

\[ \mu = \epsilon^2 - \frac{\eta}{\epsilon} + \frac{\delta^2}{4 \epsilon^2} \]  

and a Hopf bifurcation when

\[ 4c_0^4 + c_0^2 \left(1 + \frac{4\eta}{\epsilon}\right) = \epsilon^2 - \frac{\eta^2}{\epsilon^2} \]  

provided that \( -2 \frac{\eta}{\epsilon} + \frac{\delta}{\epsilon} - 4c_0^2 > 0 \). At the saddle-node bifurcation a pair of asymmetric states annihilates; the Hopf bifurcation leads to asymmetric oscillations about each of the reflection-related asymmetric states.

It is a simple matter to compute the normal form at each of the primary bifurcations. Since the transformation \( (\phi, \alpha, \beta, \gamma) \rightarrow (\phi + \pi, -\alpha, -\beta, -\gamma) \) leaves equations (3) invariant it suffices to compute the normal forms of the possible bifurcations at only one of the trivial steady states, e.g. at SS\(_0\). The normal forms for the corresponding bifurcations at SS\(_\pi\) can then be obtained by applying the above transformation.
For example, the pitchfork at $SS_0$ producing $P^\pm$ is described by

$$\dot{x} = \frac{\varepsilon^2 \mu}{\eta + \varepsilon^2} x - \frac{\varepsilon^2}{\eta + \varepsilon^2} \left(\varepsilon^2 + \frac{1}{2} \delta\right) x^3.$$  \hfill (29)

Likewise, the Hopf bifurcation of the $SS_0$ is supercritical when $\eta < \frac{1}{2}(\delta - \epsilon)$. Note that the Hopf bifurcation can make a transition from supercritical to subcritical.

These various bifurcations are organized by appropriate degeneracies in the above bifurcations. Of greatest importance are the Takens–Bogdanov bifurcations, characterized by the presence or absence of reflection symmetry. The symmetric case arises when the Hopf frequency of $SS_0$ vanishes. This occurs when $\mu = \epsilon - \delta, \eta = -\varepsilon^2$. The resulting multiple bifurcation is described by the (unfolded) normal form

$$\dot{x} = y \quad \dot{y} = (\eta_1 + \epsilon \mu_1)x + \mu_1 y - (\epsilon^3 + \frac{1}{2} \delta \eta_1)x^3 - (\frac{1}{2}(\delta - \epsilon) + 3 \varepsilon^2) x^2 y.$$  \hfill (30)

Here the unfolding parameters $\mu_1, \eta_1$ are defined by $\mu = \epsilon - \delta + \mu_1, \eta = -\varepsilon^2 + \eta_1$.

The generic Takens–Bogdanov bifurcations that occur when the asymmetric fixed points have a double zero eigenvalue take place when $\mu = 3\varepsilon^2 - \frac{1}{4}\varepsilon^2 - \frac{1}{4}, \eta = \frac{1}{2} \varepsilon^2 + \frac{1}{2} - 2\varepsilon^2$ and are described by the normal form

$$\dot{x} = y \quad \dot{y} = c_0 \left(\frac{\varepsilon^2 - \delta^2}{4\varepsilon^2}\right) x^2 + c_0 \left(1 + \frac{2\delta}{\varepsilon}\right) xy$$  \hfill (31)

subject to appropriate unfolding.

Further organization can be obtained by focusing on additional degeneracies. The degenerate pitchfork at $SS_0$ occurs at $\delta = -2\varepsilon^2$ and yields the normal form $\dot{x} = \frac{1}{4\varepsilon^2 + \varepsilon^2} x^5$, while the sign of the quintic term in the degenerate Hopf bifurcation ($\eta = \frac{1}{2} (\delta - \epsilon)$) is given by the sign of $(\epsilon - \delta) - 24 \delta \varepsilon$ subject to the restriction $(\epsilon - \delta) > 6 \varepsilon^2$. These degeneracies can be brought together, yielding the following two degenerate Takens–Bogdanov bifurcations with $Z_2$ symmetry. At $\mu = 2\varepsilon^2 + \epsilon, \eta = -\varepsilon^2, \delta = -2\varepsilon^2$ we have

$$\dot{x} = y \quad \dot{y} = \left(\frac{\epsilon}{2} - 2\varepsilon^2\right) x^2 y + \frac{\varepsilon^5}{4} x^5$$  \hfill (32)

while at $\mu = 6\varepsilon^2, \eta = -\varepsilon^2, \delta = -6\varepsilon^2$ we have

$$\dot{x} = y \quad \dot{y} = \epsilon^2 (2\epsilon - \frac{1}{2}) x^3 + 3\epsilon^2 (2\epsilon - \frac{1}{2}) x^4 y.$$  \hfill (33)

Finally, the Hopf bifurcation at $SS_0$ becomes doubly degenerate when $\mu = \epsilon - \delta, \eta = \frac{1}{6} (\delta - \epsilon), \epsilon = \delta (1 + 24 \varepsilon)$. A long calculation shows that in both cases the seventh-order term is positive. All of the above singularities occur at arbitrarily small values of the imperfection parameters $\epsilon, \delta, \eta$, and hence are realizable in the unfolding. Additional (codimension 4) singularities are also possible, but occur away from the origin.

The local bifurcations analysed above are all familiar from bifurcations on $\mathbb{R}^2$, and consequently ignore the fact that the phase space of the present problem is isomorphic to the cylinder $\mathbb{R} \times S^1$. This fact implies that another class of oscillations is possible, for which the phase $\phi$ continually increases (cf Sanders and Cushman [35]). In the following such oscillations will be referred to as TW, to distinguish them from standing waves (SW), for which $\phi$ oscillates between a maximum and a minimum. All the oscillations discussed thus far are of SW type. Since we know that at large $\mu$ the present system does contain a pair of such TW, we need to understand in detail the transitions that lead to such waves. This is done most easily by using averaging, as described in sections 5 and 6.
Considering $\mu$ as a distinguished bifurcation parameter and ($\epsilon, \delta, \eta$) as imperfection parameters, a variety of distinct bifurcation sequences can be observed for different choices of ($\epsilon, \delta, \eta$) when $\mu$ is varied. These bifurcations are to some extent organized by the degenerate bifurcations discussed above, although the bifurcations to TW, being global in nature, are not captured by the local degeneracies. In Knobloch et al [23] the different phase portraits for a particular bifurcation sequence ($\delta = 0.3, \eta = 0.02, \epsilon = 0.2$) are presented. A unified, qualitative description of the individual bifurcations occurring in this sequence is shown in the bifurcation diagram of figure 1, where the average over a period of $|\phi|$ (figure 1(a)) and $\phi$ (figure 1(b)) is sketched for the various solutions as a function of $\mu$. The observed sequence of bifurcations with increasing amplitude is as follows. First, asymmetric steady states $P^\pm$ are created from $SS_0$ and terminate on $SS_\pi$ in pitchfork bifurcations. On the $P^\pm$ branch an unstable asymmetric periodic orbit $SW_a$ is born and changes into a symmetric periodic orbit $SW_s$ in an ambiclinic (figure of eight) orbit, analogous to the global bifurcation that occurs in the unfolding of the Takens–Bogdanov bifurcation with $Z_2$-symmetry (e.g. Knobloch and Proctor [21]). The $SW_s$ orbit is stabilized by a saddle-node bifurcation and in turn loses stability in a saddle-node bifurcation and then becomes homoclinic to $SS_0$ where it changes into a TW. In the phase space the homoclinic orbit connects $SS_0$ to $SS_\pi$; the terminology homoclinic is used since $SS_\pi$ is identified with $SS_0$ by the $2\pi$-periodicity of the vector field. The unstable TW state produced by this global bifurcation is stabilized via a further saddle-node bifurcation and thereafter persists to larger values of $\mu$. Thus, the overall transition is from a stable steady state $SS_0$ to a stable TW, as predicted from the perfect bifurcation ($\epsilon = \delta = \eta = 0$), but during this overall transition a number of other stable and unstable states appear which should be
observable in experiments. In addition, Knobloch et al [23] found that both the appearance of the symmetric SW and their subsequent disappearance via saddle-node bifurcations was associated with infinite cascades of heteroclinic bifurcations connecting $SS_0$ and $SS_\pi$. These bifurcations accumulate at a finite value of the bifurcation parameter (from below and above, respectively) according to a well defined law, at which the saddle-node bifurcations take place. In the following we shall show that under appropriate conditions the location of such saddle-node bifurcations can be computed analytically, but will not compute the associated cascades of global bifurcations. It should be borne in mind, however, that such cascades are inevitable, and that the saddle-node bifurcations are a signature of their presence.

In the PDE context the vectors introduced in the abstract formulation of section 2 are periodic functions of a spatial variable $x$. The parity-breaking bifurcation then gives rise to states of the general form

$$\Psi(x,t) = u_0(x + \phi(t)) + c(t)w_1(x + \phi(t)) + O(\varepsilon + c^2)$$

(34)

where $u_0(x)$ is even in $x$ and $w_1(x)$ is odd. If we retain only the leading terms in the Fourier expansion of these functions we obtain a function $\Psi$ that is still reflection symmetric. This is also the case for the commonly used representation

$$\Psi(x,t) = A \cos k[x + \phi(t)] + Bc(t) \sin 2k[x + \phi(t)]$$

(35)

even though this expression captures the tilt in the solution that one expects of TW, cf. Coullet et al [5, 12]. In order to avoid accidental reflection symmetries it is necessary to include additional harmonics. This point notwithstanding we have used representation (35) to illustrate the resulting wave patterns. In this representation the fixed point $(\phi, c) = (0, 0)$ ($SS_0$) represents a time-independent and reflection-symmetric state; in contrast, when $(\phi, c) = (0, c_0)$ (P) the solution remains time independent but acquires a tilt (figure 2(a)). When $(\phi, c)$ are both time dependent and take the form of symmetric oscillations the resulting SW is invariant under reflection followed by evolution through half a period (figure 2(a)), while $(\phi(t), c(t))$ in the form of a rotation generate a modulated TW, as shown in figure 2(b).

4. Rescaled equations

In this section we introduce the scalings of the perturbed equations (3) which allow a relatively complete analysis of the global bifurcations to both SW and TW. We distinguish two cases, a generic case in which all imperfection parameters are of the same order, and a degenerate case in which two of the imperfection parameters are small relative to the others, and are considered as ‘unfolding parameters’ of a degeneracy in the imperfection.

We begin by writing system (3) as a single second-order equation for the phase $\phi$,

$$\ddot{\phi} + a \sin \phi = \sigma (\lambda + b \cos \phi - \dot{\phi}^2) \dot{\phi} + O(\sigma^2)$$

(36)

To make (36) amenable to a perturbation analysis we introduce the scaling

$$\sigma = \sqrt{|\gamma| \varepsilon} \quad \tau = \sigma t \quad \mu = \lambda \sigma^2 \quad a = -\text{sgn}(\gamma) \quad b = \frac{\beta - \alpha}{|\gamma|}.$$  

(37)

The generic situation is that $\alpha, \beta, \gamma$ as well as $\beta - \alpha$ are $O(1)$ quantities, with the smallness of the imperfection captured solely by the overall parameter $\varepsilon$, $0 < \varepsilon \ll 1$. In this case the scaling (37) takes $\mu$ and $\varepsilon$ to be of the same order and the timescale becomes $O(\sqrt{\varepsilon})$. Equation (36) then simplifies to

$$\ddot{\phi} + a \sin \phi = \sigma (\lambda + b \cos \phi - \dot{\phi}^2) \dot{\phi} + O(\sigma^2)$$

(38)
Figure 2. Stable solutions of the system (3) in the \((x, t)\) plane using the representation (35) with \(A = 2.5, B = 2.5, k = 1.0\), showing (a) a symmetric SW, and (b) a TW, both at \(\mu = 0.1647\).

where the dot now refers to \(d/d\tau\). In (38) \(\lambda\) and \(b\) are considered to be \(O(1)\) quantities, while \(\sigma = O(\sqrt{\varepsilon})\) plays the role of a small parameter that specifies the magnitude of the dissipative perturbation of the Hamiltonian system on the left-hand side. Without loss of generality we suppose \(a = 1\) since the results for \(a = -1\) may be recovered using the transformation \(\phi \rightarrow \phi + \pi\). In the next section we analyse (38) by using the method of averaging, and give a complete analysis of the local and global bifurcations as a function of \(\lambda\) and \(b\) for \(a = 1\).

An important consequence of the scaling introduced above is the elimination of the asymmetric fixed points. These fixed points are only present in the regime where \(\gamma\) and \(\beta - \alpha\) are both \(O(\sigma)\). In this case the scaling (37) leads to

\[
\ddot{\phi} + (a - s^2 \cos \phi) \sin \phi - \sigma s g(\cos \phi) \sin \phi = \sigma (\lambda + b \cos \phi - 3s^2 \sin^2 \phi - 3s\dot{\phi} \sin \phi - \dot{\phi}^2) \dot{\phi} + O(\sigma^2) \tag{39}
\]
where
\[ \frac{s}{\alpha} = \frac{\sigma}{|\gamma|} = \sqrt{\frac{\varepsilon}{|\gamma|}}. \] (40)

\( g(\cos \phi) = \lambda + b \cos \phi - s^2 \sin^2 \phi \) describes a Hamiltonian perturbation and the dot again refers to \( d/d\tau \). Note that here \( \sigma \) and hence the timescale is of the order \( O(\varepsilon) \), \( \mu = O(\varepsilon^2) \), while \( \lambda, b \) and \( s \) are \( O(1) \) quantities. Since the left-hand side of (39) is Hamiltonian, this equation is also amenable to the method of averaging; however, the case \( a = -1 \) can only be recovered from \( a = 1 \) via \( \phi \to \phi + \pi \) if, in addition, \( b \) and \( s \) change sign. Consequently, in the following we restrict our attention to the case \( a = 1 \), but analyse local and global bifurcations for both positive and negative values of \( b \) and \( s \). The results of this analysis are described in section 6 in the \((\lambda, b)\) plane for particular choices of \( s \).

5. Averaging for the generic case

We investigate equation (38) with \( a = 1 \) first. When \( \sigma = 0 \) the equation reduces to the undamped pendulum equation with the integral
\[ \frac{1}{2} \dot{\phi}^2 + 1 - \cos \phi = h. \] (41)

For \( 0 < h < 2 \), equation (41) defines a one-parameter family of SW oscillations while for \( h > 2 \) it defines a one-parameter family of rotations with \( \phi \) monotonically increasing or decreasing. For example, the standing oscillations take the form \( \sin \frac{\phi}{2} = k \text{sn}(t, k) \) with modulus \( k^2 = \frac{h}{2} \); however, in the following we shall not need the explicit form of these solutions.

These families of periodic orbits are destroyed by the \( O(\sigma) \) terms. Being autonomous, the effects of these terms will be captured by the resulting slow evolution of the Hamiltonian \( h \). The condition that a periodic orbit, corresponding to a given value of \( h \), persists under the perturbation for \( \tau \leq O(\sigma^{-1}) \), i.e. for \( t \leq O(\varepsilon^{-1}) \), is \( \langle h \rangle = 0 \), where the brackets denote the average over a periodic orbit of the Hamiltonian system with energy \( h \). This condition can be written in the form
\[ \oint [\lambda + b \cos \phi - \dot{\phi}^2] \dot{\phi} \, d\phi = 0. \] (42)

A straightforward calculation shows that in the case of SW \( (h < 2) \) the above condition reduces to the equation
\[ (1 - k^2)[3\lambda - 12k^2 - (b - 8)] - [3\lambda - (b - 8)(1 - 2k^2)] \Psi(k) = 0 \] (43)
where
\[ k^2 = \frac{h}{2} \quad \Psi(k) = \frac{E(k)}{K(k)} \quad 0 < h < 2 \]
and \( K(k), E(k) \) are the complete elliptic integrals of the first and second kind. Equation (43) determines \( k \) and hence the ‘energy’ \( h \) of the surviving oscillations as functions of the parameters \( \lambda \) and \( b \). For \( h > 2 \) equation (42) becomes
\[ 2(2 - b)(1 - k^2) + [3\lambda k^2 + (b - 8)(2 - k^2)] \Psi(k) = 0 \quad k^2 = \frac{2}{h} < 1 \] (44)
and determines values of \( k \) and \( h \) of surviving rotations. These correspond in the original system to TW with superposed periodic modulation, and are periodic solutions of equation (38).
The persistence conditions (43) and (44) provide the starting point for determining all subsidiary bifurcations in the system (38). Differentiating (43) with respect to \( h \) yields

\[
\lambda - b + 12(1 - k^2) + 2(b - 6)\Psi(k) = 0.
\]

Equations (43) and (45) have the solution

\[
\frac{1}{2} \lambda = \frac{1 - k^2 - 2k^2\Psi(k) + (2k^2 - 1)\Psi^2(k)}{1 - k^2 + (2k^2 - 4)\Psi(k) + 3\Psi^2(k)}, \quad \frac{1}{2} b = \frac{7 - 13k^2 + 6k^4 + (14k^2 - 16)\Psi(k) + 9\Psi^2(k)}{1 - k^2 + (2k^2 - 4)\Psi(k) + 3\Psi^2(k)}
\]

parametrized by \( k \) \((0 < k < 1)\), and determine the curve in the \((\lambda, b)\)-plane along which the SW undergo a saddle-node bifurcation. Analogously, differentiation of (44) yields

\[
\lambda k^2 + b(k^2 - 2) - 2(6 - b)\Psi(k) = 0
\]

and leads to

\[
\frac{1}{2} \lambda = \frac{2 - 3k^2 + k^4 - (2k^4 - 4k^2 + 4)\Psi(k) + (2 - k^2)\Psi^2(k)}{k^2(1 - k^2 + (2k^2 - 4)\Psi(k) + 3\Psi^2(k))}, \quad \frac{1}{2} b = \frac{-1 + k^2 + (2k^2 - 4)\Psi(k) + 9\Psi^2(k)}{1 - k^2 + (2k^2 - 4)\Psi(k) + 3\Psi^2(k)}
\]

as the parametric representation of the curve of saddle-node bifurcations of TW. Hopf bifurcations from \( SS_0 \) can be recovered by expanding (43) around \( k = 0 \). Setting the coefficient of \( k^2 \) equal to zero yields \( \lambda = -b \) as the locus of the Hopf bifurcation, while the coefficient of \( k^3 \) yields the stability coefficient. In accordance with the local analysis of (38) we find that the oscillations branching off \( SS_0 \) are stable for \( \lambda < 6 \) and unstable for \( \lambda > 6 \). Finally, the locus of the homoclinic bifurcation can be determined from the limit \( h \to 2 \), either from below or from above. Taking the limit \( k \to 1^- \) in (43) and (44) yields the straight line \( 3\lambda + b = 8 \). This completes the determination of the codimension-1 bifurcations.

There are two points in the \((\lambda, b)\)-plane where bifurcations of codimension 2 occur. At \((\lambda, b) = (6, -6)\) the stability coefficient of the Hopf bifurcation vanishes resulting in a degenerate Hopf bifurcation. At \((\lambda, b) = (2, 2)\), a degenerate homoclinic bifurcation occurs in which the saddle-node bifurcations of the SW and TW approach one another.

In figure 3 we show the stability diagram formed by the bifurcation lines determined above in the \((\lambda, b)\)-plane. In addition, the structurally stable phase portraits corresponding to the open regions of the stability diagram are sketched. Note that for any fixed \( b \) there is an overall transition with increasing \( \lambda \) from region A, where \( SS_0 \) is the only attractor, to region B with a stable TW. This picture is consistent with the general observation that the influence of symmetry-breaking imperfections becomes weaker farther from the bifurcation point of the imperfect system. In figure 4 we sketch the bifurcation diagrams resulting from figure 3. In essence we have three different types of diagrams, distinguished by the number of saddle-node bifurcations. The bifurcation diagrams for \( b > 2 \) show a monotonic transition from a stable SW to a stable TW, across a homoclinic bifurcation. In contrast, for \( b < -6 \) the SW is unstable resulting in a hysteretic transition between \( SS_0 \) and TW. Hysteresis is also present for \(-6 < b < 2\) but involves SW in addition to \( SS_0 \).
6. Averaging for the nongeneric case

In this section we give a partial analysis of equation (39) with \( a = 1 \). If \( \sigma = 0 \), this equation also reduces to a Hamiltonian system. The Hamiltonian is given by

\[
h = \frac{1}{4} \dot{\phi}^2 + V(\phi)
\]

with the potential

\[
V(\phi) = (1 - s^2) \sin^2 \frac{\phi}{2} + s^2 \sin^4 \frac{\phi}{2}.
\]
The extra factor of $\frac{1}{2}$ has been introduced for convenience. The form of the potential is shown in figure 5. Observe that for $s^2 < 1$ the potential has the same qualitative form as the potential of a pendulum. In contrast, for $s^2 > 1$, $V(\phi)$ has a pair of asymmetric minima at $\pm \phi_a$, where $\cos \phi_a = \frac{1}{s}$. The asymmetric fixed points $P^\pm$ defined in section 3 thus occur in the regime $s^2 > 1$ and are created in a pitchfork bifurcation from $SS_0$ when $s$ passes through $s^2 = 1$ from below. Since we expect similar results as in the previous section when the asymmetric fixed points are absent ($s^2 < 1$), we confine our analysis to $s^2 > 1$.

In this parameter range equation (49) defines for $h > 1$ a one-parameter family of rotations, denoted by TW, and for $0 < h < 1$ a one-parameter family of (symmetric) oscillations with zero mean, denoted by SW. In addition, there is a pair of one-parameter families of asymmetric oscillations, denoted $SW_a$, around $P^\pm$ for $V_{\text{min}} < h < 0$, where $V_{\text{min}} = V(\phi_a) = -\frac{s^2 - 1}{4s^2}$. The condition that each of these types of periodic orbits persists under the dissipation is again $\langle \dot{h} \rangle = 0$ and leads to the requirement

$$\int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \sqrt{h - V(\phi)}[\lambda + b \cos \phi - 3s^2 \sin^2 \phi - 4(h - V(\phi))] \, d\phi = 0 \quad (51)$$

where $(\phi_{\text{min}}, \phi_{\text{max}}) = (0, \pi)$ for TW, $(0, \phi_0)$ with $V(\phi_0) = h, \phi_0 > 0$, for SW and $(\phi_-, \phi_+)$ with $V(\phi_\pm) = h, 0 < \phi_- < \phi_+$, for $SW_a$ (see figure 5). Note that the Hamiltonian perturbation governed by $g(\cos \phi)$ does not contribute to the persistence condition (51).
Parity-breaking bifurcation in inhomogeneous systems

Figure 5. The potential \( V(\phi) \) for \( s^2 < 1 \) and \( s^2 > 1 \).

and that this condition depends on \( s^2 \) only. Consequently in the following we restrict our attention to \( s > 0 \). Introducing \( x = \sin^2 \frac{\phi}{2} \) as integration variable, (51) becomes

\[
P(h) \equiv \int_{x_{\text{min}}}^{x_{\text{max}}} R(x) \frac{dx}{\sqrt{y}} = 0 \tag{52}
\]

with

\[
R(x) = [h - (1 - s^2)x - s^2x^3][\lambda + b - 4h + 2(2 - b - 8s^2)x + 16s^2x^2] \tag{53}
\]

\[
y = x(x - 1) \left( x^2 + \frac{1 - s^2}{s^2}x - \frac{h}{s^2} \right) \tag{54}
\]

and the new integration limits \((x_{\text{min}}, x_{\text{max}}) = (0, 1)\) for TW, \((0, x_+)\) for SW and \((x_-, x_+)\) for SW\(_a\), where

\[
x_{\pm} = \frac{1}{2s^2} \left[ s^2 - 1 \pm \sqrt{(s^2 - 1)^2 + 4hs^2} \right]. \tag{55}
\]

To determine saddle-node bifurcations we also need the derivative of the persistence integral \( P(h) \). From definition (52) we obtain

\[
2 \frac{dP(h)}{dh} = \int_{x_{\text{min}}}^{x_{\text{max}}} [\lambda + b - 12h + 2(6 - b - 12s^2)x + 24s^2x^2] \frac{dx}{\sqrt{y}}. \tag{56}
\]

Well known recursion formulae [15] can be used to reduce any integral of the form

\[
Y_n = \int_{x_{\text{min}}}^{x_{\text{max}}} x^n \frac{dx}{\sqrt{y}} \tag{57}
\]

with \( n > 2 \) to a linear combination of the standard integrals \( Y_0, Y_1, Y_2 \). Applying these recursion formulae to the individual integrals in (52) and (56) yields, after dividing by \( Y_0 \),

\[
h \left( \lambda + \frac{b}{2} - 4h + \frac{5}{3} - \frac{4}{3}s^2 \right) + \left[ h \left( \frac{14}{3} - b - 28s^2 \right) + (1 - s^2) \left( \frac{8s^2}{3} - \frac{10}{3} - \lambda \right) \right] \frac{Y_1}{Y_0} \\
+ \left( 1 + \frac{h}{2} - \lambda s^2 - \frac{14}{3}s^2 + \frac{28}{5}hs^2 + \frac{8}{3}s^4 \right) \frac{Y_2}{Y_0} = 0 \tag{58}
\]

for the persistence condition, and

\[
\lambda + b - 12h + 2(6 - b - 12s^2) \frac{Y_1}{Y_0} + 24s^2 \frac{Y_2}{Y_0} = 0 \tag{59}
\]
for its derivative. Equations (58) and (59) together lead to a parametric representation 
\((\lambda(h), b(h))\) of the locus of saddle-node bifurcations for each of the three types of periodic 
orbits. Homoclinic bifurcations involving the saddle points SS\(_0\) and SS\(_\pi\) are obtained from 
(58) in the limit \(h \to 0\) and \(h \to 1\), respectively. Note that the homoclinic connection at 
SS\(_\pi\) actually yields two symmetrically related homoclinic orbits (ambiclinic orbits) as in the 
\(Z_2\)-symmetric Takens–Bogdanov normal form (cf [21]). Finally, the Hopf bifurcation from 
\(P^\pm\) can be determined either from (58) in the limit \(h \to V(\phi_a)\) or directly from (39), leading 
to the straight line \(\lambda = 3s^2 - \frac{b+3}{s^2}\). The bifurcating periodic orbit is stable for 
\(b < -6\) and unstable for \(b > -6\), i.e. there is a degenerate Hopf bifurcation at \(b = -6, \lambda = 3(s^2 + \frac{1}{s^2})\).

The analytic expressions for the other codimension-1 bifurcations are very unwieldy and 
have not been calculated explicitly. Instead, the corresponding curves in the 
\((\lambda, b)\)-plane have been computed numerically for fixed \(s\) via (58) and (59) in the case of saddle-nodes 
and via (58) for the other codimension-1 bifurcations. For the numerical computation it is 
convenient to translate the elliptic integrals \(Y_0, Y_1, Y_2\) to Legendre’s canonical form, since 
algorithms for computing them in this form are well known [31]. We set 
\(k^2 = \frac{p^2}{2}\) for the rotations and asymmetric oscillations and \(k^2 = \frac{1}{p^2}\) for the symmetric oscillations, where

\[
p^2 = \frac{2r}{s^2 - 1 + 2h + r}, \quad r = \sqrt{(1 - s^2)^2 + 4hs^2}.
\]

The ratios \(\frac{Y_1}{Y_0}, \frac{Y_2}{Y_0}\) then take the form

\[
\frac{Y_1}{Y_0} = x_-(1 - \Sigma(\xi, k))
\]

\[
2\frac{Y_2}{Y_0} = -x_+x_- - \frac{x_-(1 - 2s^2)}{s^2} - x_+(1 - x_-)\Psi(k) + \frac{x_-(1 - 2s^2)}{s^2}\Sigma(\xi, k)
\]

for the rotations,

\[
\frac{Y_1}{Y_0} = x_-(1 - \Sigma(\xi, k))
\]

\[
2\frac{Y_2}{Y_0} = x_- - \frac{x_-(1 - 2s^2)}{s^2} - (x_+ - x_-)\Psi(k) + \frac{x_-(1 - 2s^2)}{s^2}\Sigma(\xi, k)
\]

for the symmetric oscillations, and

\[
\frac{Y_1}{Y_0} = x_-\Sigma(\xi, k))
\]

\[
2\frac{Y_2}{Y_0} = -x_+x_- - x_+(1 - x_-)\Psi(k) - \frac{x_-(1 - 2s^2)}{s^2}\Sigma(\xi, k)
\]

for the asymmetric oscillations. Here, \(\Psi(k) = E(k)/K(k)\) as in section 5, and \(\Sigma(\xi, k) = \Pi(\xi, k) / K(k)\) where \(\Pi(\xi, k)\) is the elliptic integral of the third kind, with \(\xi = -1/(1 - x_-)\).

In figure 6 we show the division of the \((\lambda, b)\)-plane by the loci of codimension-1 
bifurcations for \(s = 2\). The phase portraits corresponding to the regions marked in figure 6 
(a scan for \(b \approx -10.5\)) are schematically sketched in figure 7. When \(s\) also varies, a number 
of codimension-3 bifurcations takes place which change the structure of the \((\lambda, b)\)-plane. 
In particular, cusp points of the saddle-node lines for both SW and SW\(_a\) are generated. 
Although the nature of the codimension-3 points responsible for the emergence of the cusps 
is not clear, we assume these are the result of three-fold degenerate homoclinic connections. 
In figure 8 we show the \((\lambda, b)\)-plane for \(s = 4\). Here both saddle-node lines encounter a cusp 
as can be clearly seen in the enlarged views (figure 9). The phase portraits corresponding 
to regions (1) and (2), inside the cusps, are sketched in figure 10.
7. Discussion and conclusions

In this paper we have shown that spatial inhomogeneities can have a dramatic effect on the bifurcation from reflection-symmetric steady states to asymmetric drifting states. That this should be the case is not surprising. The inhomogeneities are expected to be most important when the pattern drifts slowly, i.e. precisely near the symmetry-breaking bifurcation that creates it. We have seen that the inhomogeneities lead to pinning of the wavetrain, and that the process by which the pattern unpins itself can be remarkably complex. Two types of pinned states have been found, reflection-symmetric ones and asymmetric ones. The scenarios that lead to the unpinning of these states depend on their symmetry, but in both
cases involve global bifurcations, whose presence is a consequence of the breaking of translation invariance. Several such bifurcations, involving both SW and TW, have been identified, and their role in the transition to drifting states elucidated. That forced symmetry breaking of the type considered here should lead to global bifurcations is not surprising, and is in fact a natural mechanism for creating complex dynamics in higher-dimensional
systems. It suffices to cite here the work of Guckenheimer [16] and Kirk [19, 20] on the breaking of the $S^1$ symmetry in Hopf–steady-state interactions, and the work of Dangelmayr and Knobloch [7, 9] and Hirschberg and Knobloch [17] on the effects of breaking translation symmetry in the Hopf bifurcation with $O(2)$ symmetry. The experiments of Simonelli and Gollub [37] on the Faraday system in square and nearly square containers provide a vivid illustration of the importance of even small symmetry-breaking effects for the generation of complex dynamics.

The system studied in this paper is simpler, in the sense that chaotic dynamics cannot occur. Nonetheless, the various scenarios that we have uncovered, such as the infinite cascades of heteroclinic bifurcations [23], involve complex sequences of bifurcations, and would lead in higher dimensions to chaotic dynamics. Indeed, Glendinning [11] pointed
Figure 10. Phase portraits corresponding to the regions (1) and (2) marked in figure 9.

Figure 11. The time-averaged TW speed $v$ (small symbols) and imperfection-induced RMS oscillations $\Delta v$ about $v$ (diamonds) versus Rayleigh number near the termination of the TW branch from [30]. The broken horizontal line indicates the resolution limit; points falling below it are regarded as stationary convection. Full (open) symbols indicate decreasing (increasing) Rayleigh number. Triangles denote positive direction TW with $k = 3.28$; squares, negative direction TW with $k = 3.28$; circles, negative direction TW with $k = 3.18$. The TW are found both below and above the TW instability threshold computed by perturbation theory [3] (full curve) and numerical simulation [2] (broken curve). Courtesy of D Ohlsen.

out that a weak breaking of the remaining reflection symmetry retained in (3) leads to yet further types of complex behaviour.

Our analysis shows that, if all three symmetry-breaking coefficients are of the same order of magnitude, the pinned states are symmetric and the transition from them to the drifting states is relatively simple, and occurs via a Hopf bifurcation that leads to oscillations about the pinned state, followed by a global bifurcation that introduces a net drift of the pattern (cf figure 2). Asymmetric (i.e. tilted) pinned states are only present near a certain
degeneracy in the parameter space, as discussed in detail in section 6. For these the unpinning scenario is much more complex, and is characterized by infinite cascades of global bifurcations. Although the averaging methods used here to describe the dynamics of equations (3) require the presence of small parameters, we have shown elsewhere [23] that the conclusions drawn from the analysis hold in a much larger region of parameter space. Although the experimental situation is unclear, the experiments of Ohlsen et al [29, 30] on binary fluid convection in an annular container reveal a transition from steady-state convection to TW convection that is not sharp, even though three-dimensional effects are apparently absent. For larger $\mu$ the signature of the $\sqrt{\mu}$ dependence of the phase speed characteristic of this transition in a translation-invariant system is present. Near $\mu = 0$, however, this signature is washed out (see figure 11) and slowly travelling patterns are present even for some $\mu < 0$. These patterns travel in fits and starts, with long periods of stasis separating propagation. Such behaviour is typical of a TW near a homoclinic bifurcation to a reflection-symmetric state, and suggests that spatial inhomogeneities are responsible for the observed breakdown of the pitchfork bifurcation to TW.

A number of other applications may be envisaged. In addition to the parity-breaking bifurcations that occur in experiments on directional solidification [36, 10], viscous fingering [33] and flame fronts [13], closely related bifurcations also occur in two-dimensional Rayleigh–Bénard convection and its variants. Here a finite amplitude roll pattern may undergo a reflection-breaking steady-state bifurcation causing the pattern to tilt; associated with this tilt is a large-scale shear flow. This instability has been studied both in systems with up-down symmetry (Howard and Krishnamurti [18], Matthews et al [26]) and in its absence (Proctor et al [32], Lantz and Sudan [25]). In the former case the presence of the additional reflection symmetry prevents the expected drift of the tilted rolls. However, in the absence of the up-down symmetry, for example due to different boundary conditions at the top and bottom or compressibility (the latter case), the corresponding bifurcation will lead to drift along the group orbit; in this case spatial inhomogeneities in the forcing will result in dynamical behaviour of the type identified in this paper.

Acknowledgments

This work was completed while EK held a Visiting Fellowship at JILA, University of Colorado, Boulder, and was supported in part by the National Science Foundation under grant No DMS-9406144. GD and JH acknowledge support from the Deutsche Forschungsgemeinschaft.

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