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Dynamics of Travelling Waves in Finite Containers.

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Abstract. – A multiscale expansion is used to show that distant sidewalls can cause a travelling wave to reverse periodically its direction of propagation. These reversing states are two-frequency waves and appear via a secondary Hopf bifurcation from a pattern of counterpropagating waves. With increasing Rayleigh number the reversal period diverges and the reversals may become chaotic, before a hysteretic transition to nonreversing waves takes place. The predictions are in qualitative agreement with existing experiments.

The discovery of travelling-wave convection in binary fluid mixtures in finite containers [1] has raised a number of interesting theoretical issues. In systems that are horizontally translation-invariant the appearance of spatially periodic travelling waves (hereafter TW) is described well by the Hopf bifurcation with O(2) symmetry [2]. The theory provides conditions under which TW are preferred over the competing standing waves (hereafter SW) and is in reasonable agreement with the experiments [3]. A pure TW pattern cannot exist in a finite container, however. The sidewalls, in addition to modifying the TW pattern as described below, are also responsible for the presence of the so-called «blinking» states, first discovered numerically [4] and subsequently observed in experiments [5]. In these states the direction of propagation of the wave reverses periodically, the reversals becoming more and more irregular with increasing Rayleigh number R.

In this letter we describe how the presence of distant sidewalls changes the bifurcation problem, and focus in particular on the dynamics of the «blinking» states. The analysis is restricted to small values of the supercriticality parameter \( R - R_c \) and is asymptotically exact. To make the analysis systematic, we link \( R - R_c \) to the aspect ratio of the system and consider two-dimensional TW convection in the container \( \{ -L/e \leq x \leq L/e, 0 \leq z \leq 1 \} \), where \( R - R_c = O(\varepsilon^2), \ L = O(1) \), and \( 0 < \varepsilon \ll 1 \). The resulting system is described by the scaled amplitude equations [6]

\[
A_r = \{ D \partial_x^2 + s \partial_x + A + aB |B|^2 + b(A^2 + |B|^2) \} A, \tag{1a}
\]

\[
B_r = \{ D \partial_x^2 + s \partial_x + \bar{A} + a|A|^2 + \bar{b}(|A|^2 + |B|^2) \} B, \tag{1b}
\]
where \( X = \varepsilon x, \tau = \varepsilon^2 t \) are slow spatial and temporal variables, \( A \) and \( B \) are, respectively, the (complex) amplitudes of left- and right-travelling waves, and \( D \) is a complex diffusion coefficient. The quantity \( \Lambda \) is also complex and represents the threshold and frequency shifts due to the distant boundaries. In the following we use the subscripts \( r \) and \( i \) to denote real and imaginary parts. The parameter \( s \) is proportional to the group velocity, \( s \equiv (1/\varepsilon)(\partial \omega/\partial k)_{\omega} \), and hereafter is assumed to be \( O(1) \). This assumption restricts the validity of (1) to the vicinity of the codimension-two point \([7,8]\). Provided \( a_r \neq 0, b_r \neq 0, \) and \( a_r + 2b_r \neq 0 \) the complex coefficients \( a, b \) are given at leading order in \( \varepsilon \) by their values for spatially periodic solutions of the \textit{unbounded} system. In the following we assume that \( a_r < 0, b_r < 0 \) so that in the unbounded system the TW bifurcate supercritically and are stable. This condition also holds near the codimension-two point \([3]\). On the basis of symmetry arguments the boundary conditions on \( A \) and \( B \) take the form \([7,9]\)

\[
\begin{align*}
A - \varepsilon(\mu A_X + \nu B_X) &= 0, \quad B - \varepsilon(\mu B_X + \nu A_X) = 0, \quad \text{at } X = L, \\
A + \varepsilon(\mu A_X + \nu B_X) &= 0, \quad B + \varepsilon(\mu B_X + \nu A_X) = 0, \quad \text{at } X = -L,
\end{align*}
\]

where \( \mu, \nu \) are two complex reflection coefficients that can, in principle, be computed as in \([10]\). These coefficients depend not only on the nature of the walls at \( X = \pm L \), but also on \( L \).

In an unbounded system the conduction state \( A = B = 0 \) loses stability when \( \Lambda_r = 0 \); for the finite system we let

\[
A = \varepsilon^{12} A_0 + \varepsilon^{32} A_1 + \ldots, \quad B = \varepsilon^{12} B_0 + \varepsilon^{32} B_1 + \ldots,
\]

\( \Lambda - \Lambda_s = \varepsilon \Delta \) and introduce the superslow time \( T = \varepsilon \tau \). The \( O(\varepsilon^{12}) \) problem then yields the instability threshold

\[
\Lambda_s = D \left( \frac{\pi}{2L} \right)^2 + \frac{1}{4D} \left( \frac{s}{2} \right)^2,
\]

together with the corresponding eigenfunctions \( A_0, B_0 \). The streamfunction for the flow then takes the form

\[
\Phi(x, z, t) = \text{Re} \varepsilon^{32} \left\{ \nu(T) \exp \left[ -\frac{sX}{2D} \right] \exp \left[ i \omega \tau \right] + \right. \\
+ \left. w(T) \exp \left[ \frac{sX}{2D} \right] \exp \left[ -i \omega \tau \right] \right\} \exp \left[ ikz \right] \cos \frac{\pi X}{2L} f(z) + O(\varepsilon^{52}),
\]

where \( f(z) \) is a suitable eigenfunction in the vertical and \( (\nu, w) \) are complex amplitudes whose superslow evolution is determined at \( O(\varepsilon^{32}) \) from the appropriate solvability conditions for \( A_1 \) and \( B_1 \). One finds \([11,12]\)

\[
\nu_T = (\lambda + K \alpha |\nu|^2 + K b (|\nu|^2 + |w|^2)) \nu + d\bar{w}
\]

\[
w_T = (\lambda + K \alpha |\nu|^2 + K b (|\nu|^2 + |w|^2)) w + d\bar{v},
\]

where

\[
\lambda = \Delta - \frac{D_r^2}{4L^3} (\mu + \bar{\mu}),
\]
\[ d = -\frac{D\pi^2}{4L^3}(\nu \exp[sL/D] + \bar{\nu} \exp[-sL/D]), \]  
(7b) \[ K = \frac{3\pi^4 \sinh(q)}{q(q^2 + \pi^2)(q^2 + 4\pi^2)} \]  
(7c) with \( q \equiv sL_x/|D|^2 \). These equations, first discussed in [13], provide a complete description of the slow dynamics of the critical modes. The presence of the endwalls implies \((\mu, \nu) \neq (0, 0)\) and hence \( d \neq 0 \). There are then four types of solutions:

i) The trivial solution, \( v = w = 0 \).

ii) Standing waves, for which \(|v| = |w| \neq 0\). There are two types, depending on whether the flow is in phase at opposite ends of the container or out of phase. We refer to these as \( \text{SW}_{0,\pi} \), the subscript indicating the value of \( \theta = \arg v + \arg w = 0, \pi \). These two solutions are the only ones to bifurcate from the trivial state, and they do so in succession, with the first branch stable. Note that when \( s > 0 \) the left-travelling wave peaks in the left half of the box while the right-travelling wave peaks in the right half. This is responsible for the «chevron» patterns obtained from (5) as shown in the \((x, t)\)-plane (fig. 1a)). The solutions are called \( \text{SW}' \) since they look like standing waves when the amplitudes \( A_0, B_0 \) are uniform in \( X \).

iii) Travelling waves, for which \(|v| \neq |w|\). There are two types, right- and left-travelling waves, according to whether \(|v| < |w|\) or \(|v| > |w|\), and they bifurcate in a secondary pitchfork bifurcation from one of the \( \text{SW}_{0,\pi} \) branches. We call them \( \text{RTW}' \) and \( \text{LTW}' \), respectively, the prime indicating that in a finite container they cannot be pure rotating waves. They are, however, periodic in time (fig. 1b)).

iv) «Blinking» states (hereafter \( \text{MW}' \)), for which \(|v|, |w| \) oscillate periodically in time, so that in one half-period \(|v| > |w|\) and the wave travels mostly to the left while in the other

![Fig. 1.](image)

The total amplitude \( M = (|A_0|^2 + |B_0|^2)^{1/2} \) and the streamfunction \( \psi(x, z, t) \) at fixed \( z \) corresponding to a) \( \text{SW}_{0} \) for \( \lambda_x/|d| = 0 \), and b) \( \text{RTW}' \) for \( \lambda_x/|d| = 2.5 \) in a system with perfectly insulating sidewalls. The parameters used are \( a = -3, b = -1, D = 1, s = 1.2, k_x = \pi/\sqrt{2}, \omega_c = 2.0, \) and \( \varepsilon = 0.35 \). Using \( \nu = -(1/18\pi)(16 + 13\sqrt{2}i) \exp[-i\sqrt{2}nL] \) (from [10]) the length \( L \) is chosen so that \( \alpha = 0.6545 \).
$|v| < |w|$ and the wave travels mostly to the right (fig. 2a). These states are quasi-periodic and bifurcate from a SW' in a secondary Hopf bifurcation.

The conditions on $\alpha = \arg d$ for the MW' state to be present and its subsequent fate as $\lambda_r$ increases have been analyzed in detail [14]. Figure 3 shows the resulting bifurcation diagrams for $a = -3$, $b = -1$, $D = 1$ and several values of $\alpha$. The case studied by Cross [9] ($\mu$, $\nu$ real and $O(1)$) is described by fig. 3a). Here the SW' branch loses stability to a supercritical TW' and no MW' are present. This is in contrast to the «blinking» states found by numerical integration of (1, 2) for the case $\epsilon v = O(1)$ [9]. The analysis summarized in fig. 3 makes the following qualitative predictions:

a) If the experiments are carried out in the region $a_r < 0$, $b_r < 0$ (i.e., sufficiently close to the codimension-two point [3]) one of the SW' should be the first stable small-amplitude

![Fig. 2. - The total amplitude $M$ and the streamfunction $\psi(x, z, t)$ at fixed $z$ for the MW' state. a) $\lambda_r/|d| = 1.5$, and b) $\lambda_r/|d| = 1.9476$. Here $\alpha = 1.1781$ and the remaining parameters are as for fig. 1.](image-url)
state to be observed as the Rayleigh number increases. With increasing $R$ it should lose stability to $TW'$ or $MW'$. The reversal frequency near the onset of $MW'$ is $O(e^{d|d|})$.

b) The sequence of transitions and especially the existence of the $MW'$ should exhibit a strong dependence on the quantity $2L \mod 2\pi/e$. In particular, with increasing aspect ratio the reversing states should repeatedly come and go.

c) The transition between $MW'$ and $TW'$ is either via a tertiary Hopf bifurcation [14], or via the formation of a heteroclinic orbit at $\lambda = \lambda_h$ (fig. 3). In the latter case the period between reversals approaches infinity (see fig. 2b)) as $-\ln(\lambda_h - \lambda)$ and the transition to $TW'$ is hysteretic.

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Fig. 3. – Bifurcation diagrams for $\alpha = 0.2618, 0.6545, 1.1781$ and parameters values used in fig. 1. Solid (broken) lines indicate stable (unstable) solution branches.

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These conclusions have to a large extent been verified in the experiments reported in [15,16]. In particular, stable $SW'$ are found in [15], followed by a transition to stable $MW'$. Both the amplitude and the reversal period are found to increase with the Rayleigh number [15,16], and the dependence of the $MW'$ state on the aspect ratio is very sensitive [16]. Even the observation [16] of a transition with increasing Rayleigh number from periodic to chaotic «blinking» states was anticipated theoretically. In [13] we pointed out that the time-translation symmetry $T: (v, w) \rightarrow (v \exp[i\varphi], w \exp[-i\varphi])$ present in the amplitude equations (6) is expected to be broken by higher-order terms because $T$ is only a normal form symmetry and not a physical symmetry. Higher-order terms that break $T$ are expected to lead to chaotic dynamics near the global bifurcation at $\lambda = \lambda_h$. We model this effect by adding the terms $\delta(v, w)$ to the right side of (6). For small $\delta$ the two frequencies on the $MW'$ torus typically lock, in the manner described by the perturbed circle map $\varphi \rightarrow \varphi + \Omega + k \sin \varphi$, where $\Omega$ is the frequency ratio and $\varphi = \arg v - \arg w$. The locked states are created in saddle-node bifurcations as $\Omega$ varies. As $\lambda_r$ increases the torus appears to break down, although the periodic orbits continue to exist, and form their own homoclinic or heteroclinic orbits between $LTW'$ and/or $RTW'$ interspersed with regions of chaos, much as described for a related problem in [17]. In fig. 4 we show solutions corresponding to fig. 2 when $\phi/d = 0.29 \exp[0.8i]$, including a quasi-periodic solution, a frequency-locked state ($\Omega = 2/3$), and the corresponding chaotic state. The nature of the resulting reversals between $LTW'$ and $RTW'$ is indicated in fig. 5. The tendency towards irregular or chaotic reversals with increasing Rayleigh number, observed in the experiments [16], may be related to the above scenario.

In this letter we have examined the effects of distant sidewalls on the transition to travelling-wave convection. We found that the initial instability always takes the form of a pair of counterpropagating waves with the symmetries of a standing wave. These $SW'$ states are stable in a finite domain even in parameter regimes in which $TW$ states are predicted in the unbounded system. If this is the case, the $SW'$ states lose stability with increasing Rayleigh number to travelling states ($TW'$) that are a mixture of left- and right-
Fig. 4. — Phase portraits in the (Re $v$, Im $v$)-plane obtained from eq. (6) with additional symmetry-breaking terms $\delta(w, v)$. a) Quasi-periodic orbit ($\lambda_r/|d| = 1.944$), b) 2:3 phase-locked state ($\lambda_r/|d| = 1.943$) and c) chaotic state ($\lambda_r/|d| = 1.944$) with $\delta/|d| = 0.01 \exp[0.8i]$ for a) and $0.29 \exp[0.8i]$ for b), c).

travelling waves in which one or other wave dominates. Under certain conditions on the aspect ratio of the container, a “blinking” state may appear at a secondary Hopf bifurcation from a SW’ state. This state has no counterpart in the unbounded system. With increasing Rayleigh number the “blinking” becomes irregular before giving way to a nonreversing state. These results, summarised in the bifurcation diagrams of fig. 3, are in excellent qualitative agreement with existing experiments as well as providing a simple explanation for some of the larger-amplitude numerical solutions of eqs. (1) obtained in [9]. Because the above analysis is based solely on symmetry arguments (cf. [13, 14]) it applies equally well to other systems in which translation symmetry is broken weakly by endwalls. A typical example is provided by the Taylor-Couette system with counterrotating cylinders [12].

Fig. 5. — The streamfunction $\psi(x, z, t)$ at fixed $z$ corresponding to fig. 4(b), c)). In a) the reversals between LTW’ and RTW’ are periodic; in b) they are aperiodic.
periodic boundary conditions in the axial direction the Hopf bifurcation from Couette flow typically gives rise to stable spiral vortex flow (TW) and unstable ribbons (SW). We have shown, that, if this is the case, then in a finite system counterpropagating spirals (SW') will be the first state observed, followed by a distinct transition to either end-modified unidirectional spirals (TW') or, depending on the aspect ratio, to alternating spiral vortex flow (MW'). Similar behaviour is expected of the oscillatory instability of convection rolls [18]. The effect of imperfections on flow-induced oscillations in tubes is also described by the present theory [19].

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