Nearly Inviscid Faraday Waves in Slightly Rectangular Containers

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In the weakly inviscid regime parametrically driven surface gravity-capillary waves generate oscillatory viscous boundary layers along the container walls and the free surface. Through nonlinear rectification these generate Reynolds stresses which drive a streaming flow in the nominally inviscid bulk; this flow in turn advects the waves responsible for the boundary layers. The resulting system is described by amplitude equations coupled to a Navier-Stokes-like equation for the bulk streaming flow, with boundary conditions obtained by matching to the boundary layers, and represents a novel type of pattern-forming system. The coupling to the streaming flow is responsible for new types of secondary instabilities of standing waves leading to chaotic dynamics, and in appropriate regimes can lead to the presence of relaxations oscillations. These are present because in the nearly inviscid regime the streaming flow decays much more slowly than the waves, and resemble a class of oscillations discovered by Simonelli and Gollub [J. Fluid Mech. \textbf{199} (1989), 349] in a domain with an almost square cross-section.

§1. Introduction

In recent work on parametrically driven Faraday waves we have shown that in the presence of small viscosity (\(C_g \equiv \nu(gh^3 + Th/\rho)^{-1/2} \ll 1\)) the waves couple to a streaming flow driven in oscillatory viscous boundary layers at rigid walls and the free surface.\textsuperscript{1)} This flow in turn advects the waves responsible for the oscillatory boundary layers. While the detailed description of this feedback loop (and the derivation of the asymptotically exact equations) is involved, it is known\textsuperscript{1), 2)} that this coupling is responsible for different types of drift instabilities of the waves, instabilities that have been observed in experiments in annular containers\textsuperscript{3)} but are absent from the theory when the coupling to the streaming flow is neglected. Subsequent work by Higuera et al.\textsuperscript{4)} pointed out that the influence of the streaming flow is greatly enhanced for Faraday waves that are not of standing wave type; waves of this type are readily generated when the shape of the container is perturbed from one of high symmetry to one of lower symmetry. Such a perturbation produces a competition between almost degenerate modes, and introduces secondary bifurcations into the system that destabilize the primary standing waves already at small amplitude. The resulting dynamics can therefore be described by weakly nonlinear theory. Of particular interest is the presence of relaxation oscillations\textsuperscript{5)} resembling the ‘bursters’ studied in the context of pancreatic \(\beta\)-cells or models of firing neurons.\textsuperscript{6)} Relaxation oscillations of this type are unusual in fluid mechanics,\textsuperscript{7)} and this alone makes the nearly inviscid Faraday system of great interest.

The motivation for the work reported here is provided by the experiments of Simonelli and Gollub\textsuperscript{8)} who studied in detail the interaction of Faraday modes (3, 2)
and (2, 3) in square and almost square containers. In a square container these modes are related via a reflection in the diagonal and hence bifurcate simultaneously from the flat state. Abstract theory shows that a mixed mode state, consisting of an equal amplitude superposition of these two states, bifurcates at the same point. Both these states are periodic in time, and in the square container Simonelli and Gollub found no other states near the primary bifurcation. The situation becomes quite different in a rectangular container. In a container of horizontal aspect ratio 1.07 Simonelli and Gollub found dramatic relaxation oscillations near the mode interaction point, and these could be periodic or chaotic. These oscillations consisted of a long period oscillation (up to 2 hr) in the amplitudes of the (3, 2) and (2, 3) modes, a period much longer than the period of the modes themselves (0.07s). To our knowledge no detailed explanation for this remarkable behavior has been provided, although several suggestions have been made. In the present paper we explore the possibility, following Higuera et al.,
that the observed relaxation oscillations are a consequence of the coupling of the two modes with a streaming flow driven by Reynolds stresses generated in oscillatory viscous boundary layers.

§2. Coupled amplitude — streaming flow equations

The equations derived by Higuera et al.\textsuperscript{4)} for nearly inviscid Faraday waves in a rectangular but almost square domain are

\[
\dot{A}_\pm(\tau) = -[1 + i(\Gamma \pm \Lambda)]A_\pm + i\mu \bar{A}_\pm + i(\alpha_1|A_\pm|^2 + \alpha_2|A_\mp|^2)A_\pm + i\alpha_3 A_\mp^2 \bar{A}_\pm \\
\pm \gamma \int_{\Sigma} \int_{-1}^{0} \mathbf{u} \cdot g d\mathbf{x} A_\mp, \\
\frac{\partial \mathbf{u}}{\partial \tau} - [\mathbf{u} + G(A_+, A_-)] \times (\nabla \times \mathbf{u}) = -\nabla p + Re^{-1} \Delta \mathbf{u},
\]

where \(A_\pm\) are (complex) amplitudes of two eigenfunctions that are related by reflection in the \(xy\) diagonal of the square \(\Sigma\), e.g., the (3, 2) and (2, 3) spatial modes of the system, and \(\mathbf{u} \equiv (u_1, u_2, u_3)\) denotes the (incompressible) streaming flow (\(\nabla \cdot \mathbf{u} = 0\)). When the contact line is free the coefficients \(\alpha_1, \alpha_2\) and \(\alpha_3\) can be computed from inviscid theory, and are real.\textsuperscript{11)}-\textsuperscript{14)} The quantities \(\Gamma, \Lambda\) and \(\mu\) are proportional to the detuning, the departure from square cross-section, and the forcing amplitude, respectively. The detuning takes into account the mismatch between half the forcing frequency and the natural frequency of inviscid oscillations in a square container, and includes the frequency shift due to viscosity, while \(\Lambda\) is proportional to the frequency difference \(\omega_1 - \omega_2\) between inviscid oscillations along the two axes of the container. The time has been scaled by the viscous damping time of free oscillations in a liquid of height \(h\), viz. \(\delta \equiv \gamma_1 C_g^{1/2} + \gamma_2 C_g\), where \(C_g \equiv \nu(gh^3 + Th/\rho)^{-1/2} \ll 1\), and \(\gamma_1\) and \(\gamma_2\) are known constants that depend on the excited mode. Here \(T\) denotes the surface tension, \(\rho\) is the fluid density, and \(\nu\) is the kinematic viscosity. This scaling is responsible for the appearance of the Reynolds number

\[
Re = (\gamma_1 C_g^{1/2} + \gamma_2 C_g)/C_g
\]
in the Navier-Stokes-like equation (2.2) for the streaming flow. Like all other quantities in these equations the Reynolds number is formally of order one, but can in fact be both large or small. In general, if the container is not deep and the contact line is pinned the damping of the waves is dominated by the Stokes boundary layers and \( Re \sim \gamma_1/\sqrt{C_g} \), while in a deep container both terms (i.e., \( \gamma_1 C_g^{1/2} + \gamma_2 C_g \)) contribute to the damping. Typically, for the first few modes \( \gamma_2/\gamma_1 \sim 10^2 \) so that systems with \( C_g \lesssim 10^{-4} \), such as water or silicon oils in centimeter-deep containers, have Reynolds numbers \( Re = O(1/\sqrt{C_g}) \). For example, \( C_g = 2.4 \times 10^{-4} \) in the experiment of Simonelli and Gollub, while \( C_g = 2.5 \times 10^{-6} \) in the experiment of Feng and Sethna. In these systems, therefore, the streaming is only weakly damped and hence is easily driven by time-averaged Reynolds stresses.

Equation (2.2) contains a Stokes drift term \( \mathbf{G}(A_+,A_-) \) given by

\[
\mathbf{G} \equiv i(A_+\bar{A}_- - \bar{A}_+A_-)(\mathbf{g} - \mathbf{h}),
\]

and is to be solved subject to the boundary conditions

\[
\begin{align*}
\mathbf{u} &= (|A_+|^2 + |A_-|^2)\varphi_1 + (|A_+|^2 - |A_-|^2)\varphi_2 + (\bar{A}_+A_- + A_+\bar{A}_-)\varphi_3 \\
&\quad + i(\bar{A}_+A_- - A_+\bar{A}_-)\varphi_4
\end{align*}
\]

at all rigid surfaces, and

\[
\mathbf{u} \cdot \mathbf{e}_z = 0, \quad \partial \mathbf{u}_\perp/\partial z = i(\bar{A}_+A_- - A_+\bar{A}_-)\varphi_5
\]

at the free surface \( z = 0 \). Here \( \mathbf{u}_\perp \equiv (u_1, u_2, 0) \), and the vectors \( \mathbf{g}, \mathbf{h} \), and \( \varphi_1, \cdots, \varphi_5 \) are all real and computable in terms of the components of the excited inviscid linear mode of the square system. One finds that while \( \mathbf{u} \) transforms under \( \kappa : x \rightarrow -x \) and \( \pi : x \rightarrow y \) according to

\[
\kappa \mathbf{u} = (-u_1, u_2, u_3), \quad \pi \mathbf{u} = (u_2, u_1, u_3),
\]

the vectors \( \mathbf{g} \) and \( \mathbf{h} \) transform according to

\[
\kappa \mathbf{g} = (g_1, -g_2, -g_3), \quad \pi \mathbf{g} = -(g_1, g_2, g_3).
\]

We refer to (2.7) as even/even and to (2.8) as odd/odd, and note that the vectors \( \varphi_1, \cdots, \varphi_5 \) are even/even, even/odd, odd/even, odd/odd and odd/odd, respectively. It follows that when \( A = 0 \) Eqs. (2.1) and (2.2) are symmetric with respect to the two reflections

\[
x \rightarrow -x, \quad A_+ \rightarrow -A_+, \quad (u_1, u_2, u_3) \rightarrow (-u_1, u_2, u_3)
\]

and

\[
x \rightarrow y, \quad A_+ \rightarrow A_-, \quad (u_1, u_2, u_3) \rightarrow (u_2, u_1, u_3)
\]

that generate the group \( D_4 \) of symmetries of a square; when \( A \neq 0 \) (2.10) is no longer a symmetry, and Eqs. (2.1) and (2.2) are invariant under (2.9) and

\[
y \rightarrow -y, \quad A_- \rightarrow -A_-, \quad (u_1, u_2, u_3) \rightarrow (u_1, -u_2, u_3)
\]
only.

It is important to observe that in Eqs. (2.1) and (2.2) all coefficients are formally of order one. In particular the forcing of the streaming flow remains finite even in the limit of vanishing viscosity.\textsuperscript{15, 16} The boundary conditions (2.5) and (2.6) show that its magnitude is in general of order \(|A_+|^2, |A_-|^2\), and hence of order \(\mu - \mu_0\), where \(\mu_0\) is the threshold for the onset of the Faraday instability.

### 2.1. Truncation

When \(Re(|A_+|^2 + |A_-|^2) \ll 1\) we may project Eqs. (2.1) and (2.2) onto the slowest decaying streaming flow mode that couples to the amplitudes \(A_\pm\). Since this mode is odd/odd the equations become\textsuperscript{4}

\[
X' = -(1 + i(\Gamma + \Lambda))X + i(\alpha_1|X|^2 + \alpha_2|Y|^2)X + i\alpha_3X^2 + i\mu X - \gamma vX, \\
Y' = -(1 + i(\Gamma - \Lambda))Y + i(\alpha_1|Y|^2 + \alpha_2|X|^2)Y + i\alpha_3Y^2 + i\mu Y + \gamma vX, \\
v' = \varepsilon(-v + i(X\overline{Y} - Y\overline{X})),
\]

where \(X = A_+, Y = A_-\) and \(v\) represents the amplitude of the odd/odd part of the streaming flow. The use of the variables \(X\) and \(Y\) emphasizes the similarity between this system and that discussed by Higuera et al.\textsuperscript{5}

When \(\Lambda \neq 0\) Eq. (2.12) is equivariant with respect to the group \(D_4\) generated by \(R_1: (X, Y, v) \rightarrow (-X, Y, -v)\) and \(\pi: (X, Y, v) \rightarrow (Y, X, -v)\). As a result\textsuperscript{17} two types of steady states, the pure modes \(P = (Re^{i\phi}, 0, 0)\), and the mixed modes \(Ms = (Re^{i\phi}, Re^{i\theta}, 0)\) bifurcate simultaneously from the flat state. In addition the theory reveals the presence of secondary branches of general mixed modes \(MM = (X, Y, v) = (R_-e^{i\phi}, R_+e^{i\theta}, v)\), \(XYv \neq 0\) that allow the transfer of stability between the \(P\) and \(Ms\) states. When \(\varepsilon = 0\) and the \(P\) and \(Ms\) branches bifurcate in opposite directions these secondary states can undergo a Hopf bifurcation, but no Hopf bifurcations are otherwise present.

When \(\Lambda \neq 0\) the equations are equivariant with respect to the reflections \(R_1\) and \(R_2: (X, Y, v) \rightarrow (X, -Y, -v)\) only. As a result the pure modes \(P\) are split into \(P_+ \equiv (0, R_+e^{i\phi}, 0)\) and \(P_- \equiv (R_-e^{i\phi}, 0, 0)\) with \(P_+\) invariant under \(R_1\) and \(P_-\) invariant under \(R_2\); these now bifurcate in successive bifurcations from the flat state. At the same time the \(Ms\) become \((R_-e^{i\phi}, R_+e^{i\theta}, v)\) and now bifurcate in secondary bifurcations from \(P_{\pm}\). These secondary bifurcations should not be confused with the secondary bifurcations to \(MM\) which persist when \(\Lambda \neq 0\). However, the coupling to the mean flow (\(\varepsilon \neq 0\)) introduces a new class of oscillatory instabilities as well.\textsuperscript{4} As discussed next, these have a dramatic effect on the bifurcation diagrams because they can occur on the primary \(P_{\pm}\) branches.

### §3. Results

For distilled water in a square container of depth \(d = 2.5\) cm and horizontal cross-section \(6.17\text{ cm} \times 6.17\text{ cm}\) vibrated vertically with frequency close to twice the natural frequency associated with the inviscid mode \((m, n) = (3, 2)\) inviscid theory gives \(\alpha_1 = -0.194057, \alpha_2 = 0.026513, \alpha_3 = -0.274717, 11, 14\) The change of the horizontal cross-section to \(6.17\text{ cm} \times 6.6\text{ cm}\) introduces a nonzero value \(\Lambda\)
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Fig. 1. Bifurcation sets of the pure modes $P_{\pm}$ as a function of the Euclidean norm $||(X, Y)|| = \sqrt{|X|^2 + |Y|^2}$ and the parameter $\Gamma$ for $\gamma = -0.2$. Saddle-node, symmetry-breaking and Hopf bifurcations are labeled by $SN_{\pm}, SB_{\pm,MM}$ and $H_{\pm}$, respectively. The triangles indicate codimension-two points. Takens-Bogdanov, saddle-node/Hopf and saddle-node/symmetry-breaking bifurcations are labeled by $TB, SN/H$ and $SN/SB$, respectively.

into the equations; we assume, however, that this change in aspect ratio is small enough that the nonlinear coefficients $\alpha_1, \alpha_2$ and $\alpha_3$ remain unaffected. Finally, we use viscous linear theory to estimate the decay rate of the $(3, 2)$ mode and of the associated streaming flow. We obtain $\delta = 0.001, \tilde{\delta} = 0.0001$, respectively, and scale the equations with respect to the former. Thus $\varepsilon = 0.1$ and $\Lambda = 1.6$. The coefficient $\gamma$ has not been computed. Figure 1 shows the bifurcation set for $P_{\pm}$ for $\gamma = -0.2$, i.e., the amplitude on the $P_{\pm}$ branches at which various secondary bifurcations occur, as a function of the detuning $\Gamma$.

Figure 2 shows the associated bifurcation diagrams in case (a) for four different values of the detuning $\Gamma$. Since $\Lambda > 0$ the mode $P_-$ bifurcates before $P_+$ when $\Gamma < 0$ and conversely when $\Gamma > 0$. In Fig. 2(a) both branches bifurcate subcritically, and so are initially unstable. However, $P_-$ acquires stability at a saddle-node bifurcation $SN_-$ but loses it again at a symmetry-breaking bifurcation that produces the secondary states $MM$ (Fig. 3(a)). No steady states are stable beyond this point. Since this point lies to the left of the primary instability to $P_-$ the primary instability must produce time-dependence. In Fig. 2(b) the detuning is smaller, and $P_-$ bifurcates supercritically, before losing stability to $MM$ much as in (a); cf. Fig. 3(b). In Fig. 2(c) $\Gamma > 0$ and the first instability is to $P_+$. The bifurcation to $P_+$ is subcritical but the $P_+$ branch remains unstable even above the secondary saddle-node bifurcation $SN_+$. This is a consequence of an unexpected Hopf bifurcation on the $P_+$ branch below the saddle-node bifurcation. In this case there are therefore no stable steady states near either of the primary instabilities, although $P_-$ does become stable briefly at larger amplitude, after shedding an $Ms$ branch (Fig. 3(c)). Finally, in case (d) the
Fig. 2. Bifurcation diagrams for the pure modes $P_\pm$, and the mixed modes $Ms$ and $MM$, showing the Euclidean norm $||(X,Y,v)|| = \sqrt{X^2 + Y^2 + v^2}$ as a function of $\mu$ for (a) $\Gamma = -7.5$, (b) $\Gamma = -1.2$, (c) $\Gamma = 0.2232$ and (d) $\Gamma = 8.5$, when $\gamma = -0.2$. Solid (dashed) lines correspond to stable (unstable) steady states.

Fig. 3. Enlargement of the framed regions in Fig. 2.

pure modes $P_+$ bifurcate supercritically and remain stable until a secondary Hopf bifurcation $H_\pm$. This stability interval overlaps with the stability interval of $P_-$, between $SB_-$ and $H_-$, also present in case (c). The bifurcation diagrams in Fig. 2 demonstrate dramatically the role played by the streaming flow: without this flow there would be no Hopf bifurcations on the primary branches $P_\pm$, and the behavior of the system would be quite different. Figure 3(b) shows the only example of a Hopf bifurcation that persists in absence of the streaming flow.\textsuperscript{18),19)
In Fig. 4 we show some of the dynamical behavior created in the bifurcation $H_-$ on $P_-$ in Fig. 2(d). This bifurcation is found to be subcritical; the resulting periodic orbit acquires stability at a saddle-node bifurcation, before undergoing a symmetry-breaking bifurcation, followed by a cascade of period-doubling bifurcations, the first of which is shown in Fig. 4(c) projected onto the $(\text{Re}(X), \text{Re}(Y))$ plane. This projection corresponds to that used by Simonelli and Gollub in their figure 13, and allows us to conclude that most likely the behavior in this figure is produced by a scenario similar to Fig. 4. Figure 5 shows the corresponding time series.

3.1. Chaotic dynamics near onset

Chaotic dynamics are found near onset for positive detuning in the range $\Gamma_{TB} \lesssim \Gamma \lesssim \Gamma_{SN/H}$ (see Figs. 1(a) and 2(c)). Figure 6 shows the branch of $R_1$-symmetric periodic orbits created in a secondary Hopf bifurcation at $H_+$. With the exception of small intervals of stability near the first two saddle-node bifurcations this branch remains unstable until it terminates in a heteroclinic bifurcation at a) involving the pure mode $P_+$ and the origin $O$. At this bifurcation the eigenvalues of $P_+$ are $(0.74211 \pm 0.1236i, -1 \pm 4.3591i, -3.5842)$, while those at $O$ are $(0.349326, -0.1, -0.37011, -1.6298, -2.3101)$. The connection $O \rightarrow P_+$ lies in the invariant plane $(Y,v) = (0,0)$ and is therefore structurally stable, and so is the connection $P_+ \rightarrow O$ obtained from the intersection of the two-dimensional unstable manifold of $P_+$ and the four-dimensional stable manifold of $O$. Thus near a) structurally
stable $R_1$-symmetric heteroclinic cycles are present. Near a) we also find nearly heteroclinic but $R_2$-symmetric periodic orbits (Fig. 7(a)). The corresponding branch (labeled 2) terminates after a brief interval of stability in a different heteroclinic connection (Fig. 7(b)) at b) involving $P_+$ (eigenvalues $0.74211$, $0.8068$, $-1 \pm 4.4909i$, $-3.6176$) and $O$ (eigenvalues $0.4271$, $-0.1$, $-0.2204$, $-1.7799$, $-2.4717$). This global bifurcation produces a branch of $R_1$-symmetric periodic orbits (branch 3 in Fig. 6); this branch acquires stability at a saddle-node bifurcation and remains stable almost to its termination at the heteroclinic bifurcation c) involving the points $\pm P_-$, with eigenvalues $(0.50686$, $-0.19182$, $-1 \pm 0.78252i$, $-2.41504)$. Since the leading stable and unstable eigenvectors lie in the plane $\text{Re}(X) = 0$, $\text{Im}(X) = 0$ the heteroclinic cycle $P_+ \leftrightarrow -P_-$ is almost planar, with no stable periodic orbits or chaotic dynamics nearby. In fact the chaotic dynamics observed near onset (Fig. 8(c)) are not directly related to any of the heteroclinic bifurcations described above but appear to be associated with type I intermittency just before the saddle-node bifurcation at $\mu \approx 1.762315$ (Fig. 9(c)).

Figure 10 shows two other examples of chaotic dynamics near onset, projected onto the $(\text{Re}(X), \text{Re}(Y))$ plane, while Fig. 11 shows the corresponding time series. These figures, together with Figs. 8(c) and 9(c), suggest the presence of two gluing bifurcations. Specifically, in Figs. 8(c) and 9(c) we see that $\text{Re}(X)$ and $\text{Re}(Y)$ not only both oscillate about zero, but that both approach the origin essentially simultaneously. Since the forcing of the streaming flow vanishes when $X = Y = 0$ the streaming flow decays exponentially during this phase, but jumps to a larger value as soon as $X, Y$ become nonzero again. In Figs. 10(a) and 11(a) $\text{Re}(X)$ still oscillates about zero but $\text{Re}(Y)$ remains positive. Symmetry with respect to $R_2$
indicates that a reflected solution must also exist, suggesting that the symmetric attractor in Fig. 10(a) is produced by the gluing of these two asymmetric attractors in a symmetry-increasing bifurcation, even though the parameter values shown are still far from the required global bifurcation. This conjecture is supported by Fig. 11(a) which shows that both Re(X) and Re(Y) are close to the origin at the same time. Indeed, while Re(X) drifts towards the origin, Re(Y) drifts slowly away from it, eventually triggering a large excursion in Re(Y) before an abrupt return to the origin. The streaming flow once again decays freely when both Re(X) and Re(Y) are small but picks up dramatically during each excursion from the origin. It is this increase in \( v \) that arrests the growth of Re(Y) and initiates the return to the origin. Finally, in Fig. 11(b) the time series in Re(X) is more symmetric, and the system approaches the origin during both upward and downward swings. These occur more frequently as might be expected of a larger detuning.

3.2. Relaxation oscillations near onset

In the preceding section we located chaotic dynamics between the codimension-two points TB and SN/H on the \( P_+ \) branch (Fig. 1(a)), i.e., for positive detuning \( \Gamma \). In contrast, for negative detuning we find relaxation oscillations involving both steady and periodic orbits; these are associated with the TB bifurcation on the \( P_- \)

Fig. 6. Branches of steady and periodic orbits, in terms of Euclidean norm ||(X,Y,\( v \)|| and the \( L^2 \) norm ||(X,Y,\( v \)||\_L^2, respectively, as a function of \( \mu \). Thick solid (dashed) lines correspond to stable (unstable) periodic orbits; thin dashed lines denote unstable steady states. Global bifurcations are indicated with squares: a) and b) heteroclinic connection \( P_+ \leftrightarrow O \), c) heteroclinic connection \( P_- \leftrightarrow -P_- \). Parameters are as in Fig. 2(c).
branch (Fig. 1(a)) since it is this bifurcation that is responsible for the appearance of oscillations in this regime. The relaxation oscillations are found for \(-11 \lesssim \Gamma \lesssim -3\), the values of \(\Gamma\) used by Simonelli and Gollub (see figure 9 of Ref. 8)).

Figure 12(a) shows a projection onto the \(|\{(X,Y)|L_2, v}\) plane of the steady and periodic solutions of the fast system, obtained from Eq. (2.12) by setting \(\varepsilon = 0\). The fast system admits both steady and periodic solutions, with the periodic oscillations terminating in a heteroclinic bifurcation on the unstable steady states \(U\). The leading stable and unstable eigenvalues of \(U\) at this bifurcation are \(\lambda_s = -0.999548 \pm 19.89559i\), \(\lambda_u = 10.2737\), implying that the branch of periodic oscillations undergoes an infinite number of back-and-forth oscillations as it approaches the global bifurcation (see Fig. 12(a)). In this region one finds multiple coexisting stable oscillations, as described, for example, by Glendinning and Sparrow.\(^{20}\) Intervals of
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Fig. 8. Stable solutions of Eq. (2.12) for the parameters used in Fig. 6. (a) Stable periodic orbits of the branch labeled 3 in Fig. 6 in the vicinity of the saddle-node at $\mu = 1.7232$. (b) Chaotic attractor at $\mu = 1.723215$. (c) Chaotic attractor at $\mu = 1.705$, in the vicinity of the primary instability.

Fig. 9. Time series corresponding to the attractors in Fig. 10.

stable oscillations are created and destroyed at saddle-node bifurcations; within each such interval near the global bifurcation there are cascades of period-doubling bifurcations leading to intervals of chaos. Subsidiary (multipulse) heteroclinic orbits are also present. However, despite this increased complexity in the behavior of the fast system the relaxation oscillations that take place once $\varepsilon > 0$ are relatively simple. This is because the slow drift along the branch of periodic oscillations (labeled 2 in Fig. 12(a)) takes the system to the first saddle-node bifurcation, where the system jumps to the stable steady state $S$ of the fast system; thereafter it drifts along $S$ towards larger $v$ (labeled 1 in Fig. 12(a)) and hence towards the saddle-node bifurcation on $S$, where an abrupt transition back to periodic oscillations takes place. The
Asymmetric chaotic attractors for the parameters corresponding to Fig. 1(a) between the two codimension-two points $TB$ and $SN/H$. (a) $\Gamma = 0.3125$, $\mu = 1.66$, and (b) $\Gamma = 0.33$, $\mu = 1.74$.

Time series corresponding to the attractors in Fig. 10.

resulting relaxation oscillations are shown in Figs. 12(b), (c) and 13 for two values of $\varepsilon$.

Since the drift along the branch of periodic solutions of the fast system terminates at a saddle-node bifurcation the trajectory drifts past the saddle-node bifurcation and approaches the stable steady states $S$ of the fast system without visiting a neighborhood of the unstable states $U$. It is for this reason that the relaxation oscillations in the rectangular domain take a simpler form than those found by Higuera, Knobloch and Vega in an elliptical domain. In the latter the eigenvalues of $U$ control the reinjection of the trajectory to one of a pair of $S$, a fact that is responsible for the succession of transitions between relaxation oscillations of different symmetry types. Moreover, the close approach to $U$ is also responsible for the observed drifts along $U$. In contrast, any drifts along $U$ in Eq. (2.12) would be a consequence of a canard, and consequently would be restricted to an exponentially small interval in $\mu$ as $\varepsilon \to 0$. 
Fig. 12. Stable periodic relaxation oscillations for \( \Gamma = -7.5, \mu = 6.0 \). (a) The fast system \((\varepsilon = 0)\) projected on the \(||(X,Y)||_{L^2}, v) plane. Thick solid (dashed) lines indicate branches of stable (unstable) oscillations, while thin solid (dashed) lines indicate stable (unstable) steady states. The oscillations terminate in a heteroclinic bifurcation of Shil’nikov type (indicated by \(\square\)). The slow drifts, present when \(0 < \varepsilon \ll 1\), are labeled with integers to indicate the corresponding phase in the solutions computed from Eq. (2.12) with (b) \(\varepsilon = 0.1\) and (c) \(\varepsilon = 0.01\). The arrows in (a) indicate the direction of drift as well as the location of the fast transitions.

§4. Discussion

In this paper we have examined the dynamics of parametrically driven Faraday waves in square and slightly rectangular containers on the assumption that (i) the viscosity of the liquid is small (as measured by the dimensionless quantity \(C_g \ll 1\)) and (ii) the effective Reynolds number of the streaming flow driven in oscillatory boundary layers is also small. We focused on the role of forced symmetry breaking in providing a coupling between the streaming flow and the oscillations responsible
Fig. 13. Time series corresponding to the attractors in Figs. 12(b) and (c). (a) $\epsilon = 0.1$ and (b) $\epsilon = 0.01$.

for the oscillatory boundary layers driving this flow, and demonstrated, using a truncation of the coupled amplitude-streaming flow equations, that these equations exhibit both chaotic dynamics and relaxation oscillations. The former are associated with a variety of global bifurcations although at least some appear to be due to type I intermittency. The latter are typically (quasi-)periodic and consist of slow phases interrupted by fast ‘jumps’ from one phase to another, and are possible because of the disparity in time scales for the decay of free surface gravity-capillary waves and of the streaming flow. When $C_g \ll 1$ the latter decays much more slowly, a fact that is responsible for the singular perturbation structure of the truncated system. In the present case the fast system, obtained by freezing the streaming flow, exhibits both steady states and periodic oscillations; as a result we were able to find in the slow-fast system relaxation oscillations consisting of slow drifts along (slow) manifolds of steady states and periodic orbits with fast ‘jumps’ between them. These jumps are triggered by the passage through saddle-node bifurcations.

For the computations we have used the coefficients of the nonlinear terms computed from inviscid theory, and estimates of the decay rates of the surface gravity-capillary waves and of the streaming flow, and explored the dependence of the system on the forcing amplitude and frequency (i.e., detuning) for fixed symmetry breaking as measured by the parameter $\Lambda$. Thus except for the unknown value of the
coefficient $\gamma$ that describes the coupling of the surface waves to the streaming flow all the coefficient values used represent realistic values. However, despite this effort, we have not been able to reproduce in detail the relaxation oscillations reported by Simonelli and Gollub.\footnote{8}

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