Patterns are a feature of everyday life, from spreading coffee stains on a napkin to the patterns on animal coats and the spiral arms of galaxies. The patterns that form are frequently captivating, and are the subject of numerous coffee table books – my favorite is The Algorithmic Beauty of Sea Shells by H. Meinhardt (Springer 2003). Over the years fluid dynamics has motivated much of the basic research on pattern formation, and books are still being published illustrating the variety and beauty of fluid flows (see, for example, A Gallery of Fluid Motion, edited by M. Saminy, K. S. Breuer, L. G. Leal and P. H. Steen, Cambridge University Press 2003). Although the above systems are vastly different in both their scales and their physics, they none the less share many common features. Rebecca Hoyle, a senior lecturer in mathematics at the University of Surrey, has written a book focusing on the common aspects of many pattern-forming processes. The book is intended for upper division undergraduate or beginning graduate students, and is written in an informal, even chatty style. Her book chooses three physical systems as motivation, convection in a fluid layer heated from below, the patterns formed by gravity-capillary waves on the surface of a liquid in a vertically oscillating container (Faraday waves) and the famous Belousov-Zhabotinsky (BZ) chemical reaction. The book is intended to describe the mathematical techniques used for analysing different types of patterns, both in small and in spatially extended domains. As such one will not learn fluid mechanics from the book, nor learn to appreciate the physics of the parametric instability behind the formation of Faraday waves. Neither will one learn the chemistry responsible for the BZ oscillations. This is both a plus since the details of the processes may be quite involved, and a minus since one cannot help the feeling that one’s understanding remains superficial.

The book takes as its starting point the (arguable) definition that patterns form as the result of a spontaneous symmetry-breaking instability or bifurcation. From this point of view it is natural to study patterns using techniques that take maximum advantage of the presence of symmetries. These may be present because of container geometry, as a result of modeling assumptions (large systems are often discussed using periodic boundary conditions) or mathematical manipulations used to simplify the equations near a bifurcation, and put them into so-called normal form. This is the subject of equivariant bifurcation theory. This theory is well summarized in a number of books, the best of which are still the books by M.
Golubitsky, I. Stewart and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, Vols. I, II (Springer 1984, 1988). In Chapters 2 and 3 Hoyle summarizes the essential background from bifurcation theory and group theory. In Chapter 4 she turns attention to ordinary differential equations with symmetry, and in Chapter 5 extends this theory to so-called lattice patterns, that is, the patterns that tile a plane periodically in one or two dimensions. The theory of these spatially periodic patterns is much simpler than that of general patterns in the plane, because as she makes clear the symmetry group of a lattice is compact, and only discrete wavevectors are involved in the pattern formation process. Chapter 6 presents material that is rarely found in texts: superlattice patterns, quasi-patterns, hidden symmetries present, for example, in partial differential equations in certain types of domains with Neumann boundary conditions, as well as pseudoscalar representations of the Euclidean group \( E(2) \).

The second half of the book (Chapters 7-9) focuses on extended domains and uses formal asymptotic methods, primarily multiple scale techniques, to derive envelope equations for the spatial and/or temporal modulation of patterns on large scales (these include the famous Newell-Whitehead-Segel equation, the Ginzburg-Landau equation, as well as different types of phase equations, such as the Kuramoto-Sivashinsky equation), and then uses these equations to study both the constraints on the wavenumber of stable periodic patterns, and the motion of defects in otherwise spatially periodic patterns. Much of this presentation treads the path of more advanced texts such as P. Manneville’s *Dissipative Structures and Weak Turbulence*, Academic Press (1990), or *The Dynamics of Patterns* by M. I. Rabinovich, A. B. Ezersky and P. D. Weidman (World Scientific 2000), although the emphasis here is on the use of symmetry principles to derive these equations. Unfortunately for envelope equations these techniques are not as compelling, since they seek a *local* description of the system, i.e., a description in terms of an envelope function and its derivatives. While this works in many examples, there are others in which a formal asymptotic description yields nonlocal or *integro*-differential envelope equations. In such cases Hoyle is obliged to make assumptions about the size of certain coefficients, a procedure that calls into question the applicability of a purely local description to “real” systems. Chapter 10 contains a limited discussion of spiral waves, while Chapter 11 provides a welcome summary of recent developments in phase description of large amplitude slowly-varying patterns based on the Cross-Newell equation and its adumbrations; this is a powerful approach that has so far been missing from textbooks.

Hoyle’s book is the first that includes both the group-theoretic and the more formal multiple scale techniques in a single volume. It is regrettable, however, that she has not used this opportunity to show how well these techniques can do, quantitatively, to describe real world patterns. The work of Y. Liu and R. E. Ecke (Phys. Rev. E 59, 4091, 1999) on the Eckhaus instability of wall-confined travelling
wave convection in a rotating cylinder is one example that could have been used. The beautiful computations by C. Nore et al (J. Fluid Mech. 477, 51, 2003) on the von Kármán flow (the “French washing machine”) could have been mentioned in connection with the discussion of structurally stable heteroclinic cycles in Ch. 6. The other concern that I have is that students will not learn from this book how to compute the coefficients present in the equations in terms of physical parameters, and it is precisely such calculations that must be done to relate the otherwise abstract theory to real-world experiment. For example, no mention is made of how to determine, in a given system, whether a Hopf bifurcation is sub- or supercritical. Finally, because of its chatty style the book is replete with statements that are strictly speaking incorrect: [T]he point \(x_0\) is called a sink if all the eigenvalues have strictly negative real part, a source if all the eigenvalues have strictly positive real part, and a saddle otherwise (p. 29). Likewise, the notion of a normal form (pp. 47, 140) is discussed incorrectly, and as a result the statements about the origin of time-translation invariance in envelope equations (pp. 233, 236) are misleading. It is also regrettable that key results such as the Fredholm alternative theorem (p. 214) are not stated correctly. Even Fig. 4.10 representing standard results for the steady state bifurcation with \(D_4\) symmetry is misleading. The names of Yoshizawa (p. 16), Bolton (p. 285) and Sivashinsky (pp. 289, 420) are all mispelled; other misprints could have been caught by any alert editor, who surely by now should know the difference between the Greek letter \(\nu\) and italic \(v\).

However, despite the reservations just expressed I have recommended this book to all my students. The book provides a useful starting point for those of us interested in the theory of pattern formation and is well suited as a text for a first course on pattern formation.

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