Saturation of the magnetorotational instability at large Elsasser number

B. Jamroz¹, K. Julien¹, and E. Knobloch²

¹ Department of Applied Mathematics, University of Colorado, Boulder, CO 80309, USA
² Department of Physics, University of California, Berkeley, CA 94720, USA

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The magnetorotational instability is investigated within the shearing box approximation in the large Elsasser number regime. In this regime, which is of fundamental importance to astrophysical accretion disk theory, shear is the dominant source of energy, but the instability itself requires the presence of a weaker vertical magnetic field. Dissipative effects are weaker still but not negligible. The regime explored retains the condition that (viscous and ohmic) dissipative forces do not play a role in the leading order linear instability mechanism. However, they are sufficiently large to permit a nonlinear feedback mechanism whereby the turbulent stresses generated by the MRI act on and modify the local background shear in the angular velocity profile. To date this response has been omitted in shearing box simulations and is captured by a reduced pde model derived here from the global MHD fluid equations using multiscale asymptotic perturbation theory. Results from numerical simulations of the reduced pde model indicate a linear phase of exponential growth followed by a nonlinear adjustment to algebraic growth and decay in the fluctuating quantities. Remarkably, the velocity and magnetic field correlations associated with these algebraic growth and decay laws conspire to achieve saturation of the angular momentum transport. The inclusion of subdominant ohmic dissipation arrests the algebraic growth of the fluctuations on a longer, dissipative time scale.
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1 Introduction

Accretion is a process of fundamental importance in astrophysics but can only occur in the presence of an efficient mechanism for angular momentum extraction. The current consensus is that a classical instability (Velikhov 1959; Chandrasekhar 1960), since called the magnetorotational instability (or MRI for short), is particularly effective in this respect (Balbus & Hawley 1991, 1998). The instability relies on the presence of a weak poloidal field and occurs in hot disks whenever the angular velocity, \( \Omega^* \), in the disk decreases radially outward, or alternatively, the velocity shear \( \sigma^* \equiv \dot{\Omega}^*/\dot{r}^* < 0 \). In a radially bounded domain of horizontal extent \( L^* \) the above condition is modified, and becomes \( \sigma^* < -v^*_A/2\Omega^* L^* \), where \( v_A \) is the poloidal Alfvén velocity. In the following we write this condition as \( \chi \equiv -2\Omega^* \sigma^*/v^*_A \gg 1 \). The instability, which is fundamentally axisymmetric, occurs on a dynamic time scale \( \tau_{\text{MRI}} \sim \Omega^{*-1} \) and grows by extracting energy from the background shear \( \sigma^* \). For \( \Omega^*, |\sigma^*| \gg 1 \), typical in astrophysical disks, the dimensionless inviscid stability parameter \( \chi \gg 1 \) and the instability has a small wavelength in the direction parallel to the rotation and magnetic field (hereafter, the vertical direction), and takes the form of thin sheets of matter moving alternately radially (hereafter, horizontally) inward and outward as a function of the vertical coordinate.

Accretion disks with \( \chi > 1 \) can be stabilized against the MRI by sufficient dissipation (Acheson & Hide 1973; Sano et al. 1998; Lesur & Longaretti 2007). However, in astrophysical disks ohmic, \( \eta^* \), and viscous, \( \nu^* \), dissipation are small, and the corresponding Reynolds number \( R_a \equiv |\sigma^*| L^{*2}/\nu^* \) and magnetic Reynolds number \( R_m \equiv |\sigma^*| L^{*2}/\eta^* \) are both very large. MHD simulations in shearing box geometries (Balbus & Hawley 1991; Balbus & Hawley 1998) suggest that, in the absence of so-called parasitic (i.e., three-dimensional) modes (Goodman & Xu 1994), the MRI proceeds essentially unhindered by dissipation, even though some reconnection events are observed (but not resolved). More recent simulations suggest, however, that effects of dissipation are not always negligible (Sano et al. 1998; Fromang & Papaloizou 2007a; Fromang & Papaloizou 2007b) if properly resolved, and may lead to saturation of the MRI even in the absence of secondary three-dimensional instabilities.

In this article we are interested in the saturation of the MRI in the astrophysically relevant regime \( R_m \gg 1, \chi \gg 1 \). These nondimensional parameters can be combined to form the Elsasser number \( \Lambda \equiv v^*_A/\eta^* \Omega^* = 2 \tau_{\text{MRI}}/\chi \), a parameter that is also sometimes (and incorrectly) called the magnetic Reynolds number. Evidently the Elsasser number can be large or small depending on the relative magnitude of \( \tau_{\text{MRI}} \) and \( \chi \). As shown in earlier inviscid simulat-
tions (Sano et al. 1998) this parameter plays an important role in the saturation of the MRI, with saturation found for $\Lambda = O(1)$ but apparently not for larger values of $\Lambda$. Motivated by this result we have explored elsewhere (Knobloch & Julien 2005; Julien & Knobloch 2006; Julien & Knobloch 2007; Jamroz et al. 2008) asymptotically exact reduced pdes describing the nonlinear evolution of the MRI valid for $\Lambda = O(1)$, with $Rm \sim \chi \gg 1$. The reduced pde model captures the large scale back-reaction on the imposed shear through the generation of turbulent shear stresses by the MRI. Partial saturation in both the MRI mode and the turbulent shear stresses was shown to be achieved on the time scale $\sqrt{L^2/\eta^{3/2}}$, although the final saturated state is only reached on the (much longer) dissipative time scale $L^{4/3}/\eta$.

In the present article, we investigate the regime of large Elsasser numbers, $\Lambda \gg 1$, implying that $Rm \gg \chi$. It is found via numerical simulation of reduced pdes that the evolution of the MRI is described by an initial linear phase of exponential growth in time followed by a nonlinear transition to algebraic growth ($\sim t^{1/2}$) of azimuthal magnetic and poloidal velocity fluctuations, and algebraic decay ($\sim t^{-1/2}$) of azimuthal magnetic and poloidal velocity fluctuations. Remarkably, in the algebraic regime, the velocity and magnetic correlations associated with these growth laws conspire to achieve saturation of the angular momentum transport, albeit not of the fluctuations. However, the incorporation of (sub-dominant) dissipation into the reduced model arrests the growth of the fluctuations generated by the MRI on the longer, dissipative, time scale. It is found that the simulation results can be understood by analyzing the evolution of a single-mode ‘channel solution’ identified by Goodman & Xu (1994) (see also Craik &Crimean 1986). Exact solutions of the reduced equations in the algebraic scaling regime are found, and saturation of the angular momentum transport in this regime is demonstrated. Furthermore, it is shown that the single-mode solutions represent an effective parameterization of turbulent solutions upon determination of the dominant wave mode in the box. In the presence of subdominant dissipation analytical results are found for the saturation amplitude of the channel mode.

The outline of the article is as follows. In Sect. 2 we present the governing equations for the MRI in the shearing sheet approximation. Some basic properties of the linear theory are then summarized, followed by a brief discussion of exact fully nonlinear single-mode solutions to the governing equations. In Sect. 3 the reduced pde model valid for $\Lambda \gg 1$ is derived via multiscale asymptotic perturbation theory. The results of numerical simulations of the reduced model are presented in Sect. 4 and an interpretation of the results in terms of single-mode theory is given in Sect. 5. In Sect. 6 concluding remarks are made.

2 Formulation

We consider a straight channel $-L_X/2 \leq x^+ \leq L_X/2$, $-\infty < y^+ < \infty, -\infty < z^+ < \infty$, filled with an electrically conducting incompressible fluid, and rotating about the $z$ axis with constant angular velocity $\Omega_0$. We suppose that a linear shear flow $U_0 = (0, \sigma x^+, 0)$, $\sigma > 0$, is maintained in the channel, for example, by lateral boundaries that slide in the $y$ direction with speeds $\pm \sigma x^+/2$, or through the shearing box approximation where a local expansion is performed about the global shear profile $\Omega^* (r^*)$ of a disk at $r_0^*$ with $\sigma^* = d\Omega^*(r^*)/dr^*|_{r_0^*}$. In addition, we suppose that a constant magnetic field $B_0^* = (0, B^*_{pol}, B^*_{pol})$ is present and consider $y$-independent perturbations of this state. In dimensionless form, these can be written as $u \equiv (u, v, w) = (-\psi_z, v, \psi_x)$, $b = (a, b, c) = (-\sigma_z, b, \phi_z)$ and satisfy the equations (Julien & Knobloch 2006)

$$\nabla^2 \psi_t + 2\Omega v_z + J (\psi, \nabla^2 \psi) = v_z^2 \nabla^2 \psi_z + v_b^2 J (\phi, \nabla^2 \phi) + \nu \nabla^4 \psi,$$

$$v_t - (2\Omega + \sigma) \psi_z + J (\psi, \psi) = v_b b_z + v_b^2 J (\phi, b) + \nu \nabla^2 v,$$

$$\phi_z (\psi, \psi) = \psi_z + \eta \nabla^2 \phi,$$

$$b_t + J (\psi, b) = v_z - \sigma \phi_z + J (\phi, v) + \eta \nabla^2 b,$$

where $v_a \equiv B_{pol}^*/\sqrt{\mu_0 \rho U^*}$ is proportional to the Alfvén speed associated with the imposed poloidal (vertical) magnetic field, $J (f, g) \equiv \int f \partial_t g - g \partial_t f$, and $\Omega \equiv \Omega_0^* L_X^*/U^*$, $\sigma \equiv \sigma^* L_X^*/U^*$, $\nu$ and $\eta$ represent the dimensionless rotation rate, shear, kinematic viscosity, and ohmic diffusivity, respectively. Non-dimensionalizing using a velocity scale $U^*$ and the channel width $L_X^*$. The above equations are valid provided $L_X^* \ll \min (H^*, L_1^*)$ where $H^*$ is the vertical pressure scale-height and $L_1^* \equiv |d \ln \Omega^*/dr^*|^{-1}$ is the (horizontal) length scale associated with the rotational profile at $r_0^*$. Note that $B_{pol}^*$ drops out of these equations. This is not so in an annulus, where hoop stresses are present, and the MRI becomes oscillatory (Knobloch 1992; Knobloch 1996).

For normal modes of the form $\exp (\lambda + ik x + in z)$ Eqs. (1)–(4) yield the linear dispersion relation

$$p^2 [(\lambda + \eta p^2) (\lambda + \eta p^2) + v_b^2 n^2] + 2\Omega n^2 [(\lambda + \eta p^2) \{ (2\Omega + \sigma) + \sigma v_b^3 n^2 \} = 0,$$

where $p^2 \equiv k^2 + n^2$. This equation predicts a positive growth rate for sufficiently small wavenumbers $p$ whenever $\sigma < 0, v_b \neq 0$, provided only that $\nu$, and $\eta$ are sufficiently small. Moreover, for fixed $p$, the maximum growth rate is associated with the $x$-independent mode $k = 0$.

Much of the work related to the MRI has focused on the astrophysical regime $\Omega, |\sigma| \gg v_A \gg \nu, \eta$. That is, the shear is the dominant source of energy for the instability, but the instability itself requires the presence of a (weaker) vertical magnetic field. Dissipative effects are weaker still and have often been ignored entirely. This ideal limit ($\nu = \eta = 0$) is associated with the dispersion relation

$$p^2 (\lambda^2 + v_b^2 n^2) + 2\Omega n^2 [\lambda^2 (2\Omega + \sigma) + \sigma v_b^3 n^2] = 0.$$
In this case the MRI occurs in the wavenumber range \( 0 \leq p \leq \sqrt{-2\Omega/\nu} \equiv \sqrt{\chi}, \) \( n > 0, \) where \( \chi \) is the ideal MRI stability parameter. The maximum growth rate and associated wavenumber are given by
\[
\lambda_{\text{max}} = -\frac{\sigma}{2}, \quad n_{\text{max}} = \frac{\sqrt{-\sigma(\sigma + 4\Omega)}}{2\nu}, \quad k_{\text{max}} = 0. \quad (7)
\]

To highlight the importance of dissipative processes, we compare the limit of small dissipation to the ideal limit (6) in the astrophysical regime. To this end we choose the characteristic speed \( U^* \) to be \( v^*_A \) (so that \( v_A = 1 \)), write \( (\nu, \eta) = \epsilon(\tilde{\nu}, \tilde{\eta}) \), together with \( (\Omega, \sigma) = \delta^{-1}(\tilde{\Omega}, \tilde{\sigma}) \), and explore the effect on the dispersion relation (5) for \( \epsilon, \delta \ll 1 \). From (7) we identify the corresponding scales of the remaining quantities, viz., \((\eta, \lambda) \sim \delta^{-1}(\tilde{n}, \tilde{\lambda}) \). It is important to specify the meaning of the small parameters \( \delta, \epsilon \). Since \( U^* = v^*_A \) we see that \( \eta = S^{-1}, \) where \( S \equiv v^*_A L_A/\eta^* \) is the Lundquist number, and \( \nu = P_m S^{-1} \), where \( P_m = \nu^*/\eta^* \) is the magnetic Prandtl number. Likewise \( |\sigma| = Rm S^{-1} \) and \( \Omega = (\Omega^*/|\sigma^*|) Rm S^{-1} \) and it follows that \( Rm \sim O(\epsilon^{-1}\delta^{-1}), \) \( S \sim O(\epsilon^{-1}). \) Thus \( \epsilon, \delta \ll 1 \) implies that \( Rm \gg S \gg \text{max}(1, P_m) \). Moreover, the stability parameter \( \chi \sim O(\delta^{-2}) \) implying a large spectrum of unstable modes.

On considering distinguished relations between \( \epsilon \) and \( \delta \) as \( \epsilon, \delta \to 0, \) two distinct regimes are identified. Each can be characterized in terms of the nondimensional Elsasser number, \( \Lambda \equiv v^*_A /\Omega \eta = 2Rm/\chi \sim O(\delta/\epsilon) \). The results are illustrated in Fig. 1. For \( \epsilon = o(\delta), \) we find \( \Lambda \gg 1, \) and the role of dissipation in the MRI is subdominant. The dispersion relation therefore asymptotes to the ideal limit (6), where all variables are now interpreted as rescaled and replaced by corresponding variables with the superscript "\( \ast \)" (see solid curve, Fig. 1). We stress, however, that on longer time scales the subdominant dissipative effects do enter into the description of the MRI. Indeed, we show in Sect. 3 that these effects cannot be neglected for they are ultimately responsible for the saturation of angular momentum transport by the MRI. In contrast, when \( \epsilon = O(\delta), \Lambda \sim O(1) \) and the dispersion relation asymptotes to Eq. (5) (see dashed curve, Fig. 1). The saturation of the MRI in this regime is investigated by Knobloch & Julien (2005), Julien & Knobloch (2006), and Jamroz et al. (2008).

Exact, fully nonlinear, but monochromatic solutions of Eqs. (1)–(4) are known (Craig & Criminale 1986; Goodman & Xu 1994). For the gravest mode, \( k = 0, \) they take the form
\[
\psi = \Psi(t) \cos nz, \quad v = V(t) \sin nz, \quad \phi = \Phi(t) \sin nz, \quad b = B(t) \cos nz. \quad (8)
\]

For this type of solution all nonlinear terms vanish identically, and in the absence of three-dimensional parasitic instabilities the solution grows exponentially without bound, i.e., \( (\Psi(t), V(t), \Phi(t), B(t)) \sim e^{\lambda t}. \) Thus solutions of this type exhibit no saturation, a result supported by ideal numerical simulations of the MRI with general initial conditions (Balbus & Hawley 1991, 1998). In these situations the volume-averaged turbulent shear stress,
\[
(T_{xy}) \equiv -\langle uv - v^2 ab \rangle_V = -\frac{1}{2} \langle \Psi V + v^2 \Phi B \rangle e^{2\lambda t},
\]
also grows exponentially without bound. Following Julien & Knobloch 2006, who analyze the \( \Lambda \sim O(1) \) case, we postulate that a fundamental flaw of the shearing box approximation is the omission of the feedback of this shear stress on the imposed background shear flow. In a global disk geometry, this feedback is present and does ultimately lead to saturation. Specifically, we argue that for self-consistency the shearing box formulation must include the contribution of any large scale shears generated from the MRI, in addition to the local background shear. This is precisely the shear generated by the turbulent Reynolds and Maxwell stresses produced by the MRI. Thus far, these have been neglected in all derivations and implementations of the local shearing box approximations (see Umurhan & Rege 2004, for a standard derivation of the shearing box approximation).

3 Derivation of the reduced equations

In this section we follow the asymptotic analysis of Eqs. (1)–(4) (see Knobloch & Julien 2005; Julien & Knobloch 2006; Julien & Knobloch 2007), but allow for an independent scaling of the dissipative processes, and the driving rotation and shear,
\[
(\nu, \eta) = \epsilon(\tilde{\nu}, \tilde{\eta}), \quad (\Omega, \sigma) = \delta^{-1}(\tilde{\Omega}, \tilde{\sigma}), \quad (n, \lambda) = \delta^{-1}(\tilde{n}, \tilde{\lambda}),
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The dispersion relation (5) for \( x \)-independent channel mode solutions as \( \lim |\sigma| \to \infty \). Two asymptotic regimes are identified: \( \lim_{|\sigma| \to \infty} (\nu, \eta) \cdot |\sigma| \to 0 \) such that \( \Lambda \gg 1 \) (solid curve), and \( (\nu, \eta) \cdot |\sigma| \sim O(1) \) such that \( \Lambda \sim O(1) \) (dashed curve). The values \( \tilde{n}_{\text{max}} \) and \( \tilde{n}_{\text{cutoff}} \) given by (7) are indicated by diamonds.}
\end{figure}
where $\epsilon \ll 1$, $\delta \ll 1$. We also pose a multiple scales expansion in the $x$ direction with $\partial_x$ replaced by $\delta^{-1}\partial_x + \partial_X$, where $X = \delta x$. In addition, vertical derivatives are large, $\partial_z \to \delta^{-1}\partial_z$, as are time derivatives $\partial_t \to \delta^{-1}\partial_t$. In all cases we choose the characteristic scales by taking $v_{X} \equiv 1$ although $v_3$ is retained in the equations that follow to label the terms involving the magnetic field.

In parallel with the above assumptions, we need to make further assumptions about the relative magnitude of the various fields. The rapid shear ing by the azimuthal flow suggests that we take $(\psi, \phi) \to \delta^{1/2} \epsilon^{1/2} (\psi, \phi)$ and $(v, b) \to \delta^{-1} (v, b)$. Existing results for various fields. The rapid shear ing by the azimuthal flow sug-

We find it convenient to separate the fields into mean and fluctuating components,

$$f(x, X, z, t) = \mathcal{F}(X) + f'(x, X, z, t), \quad \mathcal{F} = 0,$$

where the overbar denotes the average in both space $\langle \cdot \rangle_V$ and time $\langle \cdot \rangle_t$,

$$\mathcal{F}(X) = \langle f(x, X, z, t) \rangle_{Vt} := \lim_{\tau \to \infty} \frac{1}{\tau} \int f(x, X, z, t) dx dz dt$$

Averaging Eqs. (2) and (4) yields the mean azimuthal equations

$$\epsilon \delta \frac{\partial}{\partial X} J_x(\psi, v) = \epsilon \delta \frac{\partial}{\partial X} v_{X} J_x(\phi, v) + \epsilon \delta \partial_X^2 \psi, \quad \epsilon \delta \frac{\partial}{\partial X} J_x(\phi, b) = \epsilon \delta \frac{\partial}{\partial X} v_{X} J_x(\phi, v) + \epsilon \delta \partial_X^2 \phi,$$

where $J_x(f, g) \equiv \partial_x f \partial_x g - \partial_x g \partial_x f$. The fluctuating equations are

$$\nabla^2 \psi' = (\epsilon \delta)^{\frac{1}{2}} 2\tilde{\Omega} \psi' + \epsilon \delta \partial_X J_x(\psi, \nabla^2 \psi)$$

$$= v_{X}^A \partial_X^2 \phi' + \epsilon \delta \partial_X J_x(\phi, \nabla^2 \psi) + \epsilon \delta \partial_X^2 \psi' + \mathcal{O}(\epsilon \delta^{\frac{3}{2}} \delta),$$

(13)

$$v'_z - (\epsilon \delta)^{\frac{1}{2}} \left(2\tilde{\Omega} + \tilde{\sigma} \right) \psi'$$

$$+ \epsilon \delta \partial_X J_x(\psi, v) + \epsilon \delta \left(J_x(\psi, v) - J_x(\psi, v) \right)$$

$$= \epsilon \delta \partial_X J_x(\phi, b) + \epsilon \delta \left(J_x(\phi, b) - J_x(\phi, b) \right)$$

$$+ v_{X}^A \partial_X^2 b'$$

$$+ \epsilon \delta \partial_X^2 v',$$

(14)

$$\phi' + \epsilon \delta \partial_X J_x(\psi, \phi) = \psi' + \epsilon \delta \partial_X^2 \phi'$$

$$+ \mathcal{O}(\epsilon \delta^{\frac{3}{2}} \delta),$$

$$b'_z + \epsilon \delta \partial_X J_x(\psi, v) + \epsilon \delta \left(J_x(\psi, v) - J_x(\psi, v) \right)$$

$$= \epsilon \delta \partial_X J_x(\phi, b) + \epsilon \delta \left(J_x(\phi, b) - J_x(\phi, b) \right)$$

$$+ v_{X}^A \partial_X^2 b'$$

(15)

(16)

We now expand the variables $\psi, v, \phi, b$ in terms of the small parameters $\epsilon, \delta$ in the form $\psi = \sum_i \epsilon^{i/2} \delta^{j/2} \psi_{ij}$ with similar expressions for the other three fields. From Eqs. (13)--(16) we obtain $v_{0j} = b'_{0j} \equiv 0$ to all orders. Given this fact and that $v_{00} = 0$ from (13) at $O(\delta^{-1/2})$ and $b'_{00} = 0$ from (14) and (16) at $O(\epsilon^{1/2})$, we find that the first nontrivial balances occur at $O(1)$ in (13) and (15), and $O(\epsilon^{1/2} \delta^{1/2})$ in (14) and (16).

The reduced fluctuating equations obtained are

$$\nabla^2 \psi'_{00} = \frac{\epsilon}{\delta} J_x(\psi_{00}, \nabla^2 \psi_{00})$$

$$= v_{X}^A \partial_X^2 \phi'_{00} + v_{X}^A \epsilon \delta \partial_X J_x(\phi_{00}, \nabla^2 \psi_{00})$$

$$+ \epsilon \delta \partial_X^2 \psi'_{00},$$

(17)

$$v'_{11} = \left(2\tilde{\Omega} + \tilde{\sigma} \right) \psi'_{00} + \epsilon \delta \partial_X J_x(\psi_{00}, v'_{11})$$

$$= v_{X}^A \left( v_{11} - \partial_X \phi_{00} \phi_{00} + \epsilon \delta \partial_X^2 \psi_{00}, b'_{11} \right)$$

$$+ \epsilon \delta \partial_X^2 v'_{11},$$

(18)

$$\phi'_{00} = \epsilon \delta \partial_X J_x(\psi_{00}, \phi_{00}) = \psi'_{00} + \epsilon \delta \partial_X^2 \phi_{00},$$

$$\phi'_{11} = \partial_X \phi_{00} \phi_{00} + \epsilon \delta \partial_X^2 \phi_{00},$$

$$+ \epsilon \delta \partial_X^2 v'_{11},$$

(19)

Thus nonlinear and dissipative terms are retained at leading order when $\epsilon = O(\tilde{\delta})$ but these become subdominant when $\epsilon = o(\tilde{\delta})$. Hereafter, we consider the latter case for which $\Lambda \gg 1$.

The above asymptotic analysis captures the feedback mechanism arising through the appearance of local large-scale quantities, $\partial_X \phi_{00}$ and $\partial_X \phi_{00}$. The closure of the fluctuating Eqs. (17)–(20) requires the determination of these quantities. Nontrivial balances in the mean Eqs. (11) and (12) first arise at $O(\epsilon \delta)$ giving

$$\partial_X^2 \phi_{00} = -\partial_X (\partial_X^2 \psi_{00} + \partial_X \phi_{00} b'_{11}),$$

$$\partial_X^2 \phi_{00} = -\partial_X (\partial_X^2 v'_{11} + \partial_X \phi_{00} v'_{11}),$$

(21)

(22)

We are permitted to integrate once to obtain explicit the quantities needed to close the system,

$$\partial_X^2 v_{00} = -\partial_X \phi_{00} \phi_{00} b'_{11}$$

$$\equiv u_{00} v_{00} + \partial_X \phi_{00} b'_{11},$$

$$\partial_X^2 \phi_{00} = -\partial_X v_{00} \phi_{00} b'_{11} + \partial_X \phi_{00} v'_{11} + C$$

$$\equiv u_{00} b'_{11} - \partial_X \phi_{00} v'_{11} + C,$$

(23)

(24)

where the constant $C$ is determined by radial force balance across the channel in the saturated state (Julien & Knobloch...
Relations (23)–(24) indicate how the Reynolds, Maxwell, and mixed stresses, generated by the MRI, feed back upon the imposed shear and toroidal magnetic field. Thus, they couple the dynamics of the small scale instability to the large scale. Importantly, these relations are only available if \( \hat{\nu}, \hat{\eta} \neq 0 \), that is, this feedback is only present when dissipative effects are included, even at subdominant order. Specifically, \( \partial_X \nu_0 \) represents the contribution to the imposed shear \( \hat{\sigma} \) arising from the MRI.

The fact that \( \partial_X \nu_0 \) and \( \partial_X \hat{\sigma}_0 \) are constant implies that \( \nu_0 \) and \( \hat{\sigma}_0 \) are linear functions in \( X \). This property is identical to that possessed by the projection, via Taylor expansion, of the imposed background shear velocity profile \( \Omega(x) \) onto the local geometry and emphasizes that such terms cannot be omitted. Indeed, on defining the imposed background shear stress, \( \tilde{T}_y = \hat{\nu} \hat{\sigma} < 0 \), the compensating viscous shear stress, \( T_y = \hat{\nu} \partial_X \nu_0 \), and the averaged turbulent shear generated by the MRI,

\[
\tilde{T}_{xy} \equiv -\left(u_{00}b_{11} - a_{00}u_{11}\right),
\]

we see that \( \tilde{T}_y + \tilde{T}_{xy} \equiv 0 \). This states that the total shear stress within the shearing box is constant equal to \( T_y \). For outward transport of angular momentum, required of an accretion disk, \( \tilde{T}_{xy} < 0 \), implying that \( \tilde{T}_y > 0 \) and hence that the MRI reduces the imposed shear gradient. We can therefore evaluate the efficiency of angular momentum transport by examining the quantity \( \tilde{T}_{xy} = -\hat{\nu} \partial_X \nu_0 \) (Mignone et al. 2007). Similarly for the mixed stress terms we define

\[
M_{xy} \equiv -\left(u_{00}b_{11} - a_{00}u_{11}\right)
\]

with

\[
\overline{M}_{xy} = -\hat{\eta} \partial_X \hat{\sigma}_0 + C.
\]

Also of interest is the total energy,

\[
E = \frac{1}{2} \left[ \left(|\nabla \psi_0|^2 + v_{11}^2\right) + v_{\sigma}^2 \left(|\nabla \phi_0|^2 + b_{11}^2\right) \right],
\]

and its volume average, \( \langle E \rangle_V \).

## 4 Numerical simulations of the reduced equations

The reduced system (17)–(20) and (23)–(24) with \( C = 0 \) is solved in a box of size \( L_X \times N_z \), \( L_z \), where \( N_z \) is the number of wavelengths, \( L = 2\pi/\hat{\Omega}_{\text{max}} \), of the fastest growing linear mode (determined from Eq. 7) and \( L_X \sim \delta^{-1} \) is now the large horizontal scale upon which the imposed background shear gradients \( \hat{\sigma} \) and gradients of associated feedback responses \( \partial_X \nu_0, \partial_X \hat{\sigma}_0 \) remain constant. We choose periodic boundary conditions in the \( z \) direction to simulate a vertically infinite domain, which is appropriate when the vertical instability scale is much smaller than the vertical pressure scale-height. In the \( x \) direction, we impose periodic boundary conditions as in the shearing-sheet formulation for axisymmetric flows. Hence a spectral Fourier discretization is used in space.

Time integration is performed using the semi-implicit Runge-Kutta method developed by Spalart et al. (1991). Since dissipation is subdominant the equations are treated explicitly; in this case the method is third order accurate. Closure is obtained by averaging over the small scales as described in Eqs. (23)–(24). However, for numerical efficiency, only spatial averaging, denoted by

\[
\langle f (x) \rangle_V := \lim_{V \to \infty} \frac{1}{V} \int_V f (x, X, z, t) \, dx \, dz,
\]

is performed. For validity, this assumption demands that our integration of the reduced equations reach a statistically steady state with \( |\langle \partial_X (u_{00}, b_{00}) \rangle_V - \partial_X (\nu_0, \hat{\sigma}_0) | \sim o(1) \). This is indeed the case.

In the following, we present results from our numerical simulations of the reduced equations in the subdominant dissipation regime with \( \epsilon = o(\delta) \). The simulations are distinguished by their initial conditions. We consider channel solutions, for which the initial data are \( x \)-independent and the solution remains \( x \)-independent for all time, followed by general or multi-mode solutions with \( x \)-dependent initial data. Details of our simulation runs are given in Table 1. All numerical simulations presented in this section were performed at parameter values \( v_{\Lambda} = 1, \bar{\nu} = 1, \bar{\eta} = 1, \hat{\Omega} = 1 \) together with the local Keplerian relation \( \hat{\sigma} = -3\hat{\Omega}^2/2 \).

### 4.1 Channel mode evolution

#### 4.1.1 Single Channel Mode

Figure 2 illustrates the evolution of the single-channel channel solution

\[
\psi_0 = \Psi_0(t) \cos \hat{\nu} z, \quad v_{11} = V_0(t) \sin \hat{\nu} z,
\]

\[
\phi_0 = \Phi_0(t) \sin \hat{\nu} z, \quad b_{11} = B_0(t) \cos \hat{\nu} z,
\]
with real initial data \((\Psi_0(0), V_0(0), \Phi_0(0), B_0(0)), \hat{n} = \hat{n}_{\text{max}}\), and \(\partial_X \tau_{00} = \partial_X b \equiv 0\) in the reduced equations. A logarithmically scaled plot is also included to accentuate the exponential growth in the linear phase of this instability. For these solutions \(\tau_{xy} \equiv 0\) owing to their parity in the \(z\) direction; consequently \(\partial_X \tau_{00} \equiv 0\) (since \(C = 0\)). The figure reveals a subsequent transition to a regime of slower algebraic growth or decay of the mode amplitudes: \(t^{1/2}\) for the variables \(\Phi_0\) and \(V_0\), and \(t^{-1/2}\) for the variables \(\Psi_0\) and \(B_0\). In addition to this algebraic behavior, all variables have a superposed oscillation. This is evident in the algebraically decaying modes where the oscillation swamps the decay. Figure 3 shows the corresponding evolution of \(\langle \partial_X v_{00} \rangle_V\). Remarkably, one can see that although there is unbounded algebraic growth and decay in the solution for the MRI mode, the value of \(\langle \partial_X v_{00} \rangle_V = \partial_X \tau_{00}\) as \(t \to \infty\). Thus \(\partial_X \tau_{00}\) saturates, as does the transport of angular momentum in the box. The saturated value \(\partial_X \tau_{00}\) is determined analytically in Sect. 5.2.

### 4.1.2 Multiple Channel Modes

Allowing for several simultaneous channel modes in the vertical yields more insight into the algebraically growing MRI mode. The computational domain selected has a length of \(8L\) in the vertical, where \(L = 8\pi/\sqrt{15}\), and \(L_X \equiv \epsilon^{-1}L = 10L\) in the horizontal (see Table 1). Similar results have been observed for smaller values of \(\epsilon\). We select a uniform distribution of wavelengths spanning a range of modes from \(\hat{n} \leq 4/(N, L)\) to twice the cutoff wavelength \(L_{\text{cutoff}} \equiv 2\pi/\sqrt{\chi} = \sqrt{\chi}L/4\). The mode amplitudes are sampled from a uniform distribution with upper bound \(10^{-4}\). Figure 4 shows the time evolution of the \(z\)-profile of \(\phi_{00}\), normalized with respect to its rms value. The corresponding spectral energy profile \(E(\hat{n})\) is shown in Fig. 5, where

\[
\langle E \rangle_V = \int_0^\infty E(\hat{n}) d \hat{n}.
\]

Due to the absence of quadratic Jacobian nonlinearities at \(\Lambda \gg 1\), it is found that power is only retained in the range of unstable modes originally specified in the initial data. Moreover, although the power in these modes grows algebraically, the relative growth has an inverse relationship with the wavenumber \(\hat{n}\). Hence, at large times, the largest permissible wavelength in the box dominates. This can be seen in both figures: initially the flow evolution is dominated by modes with wavenumber \(\hat{n}\) near the maximum growth rate \(\hat{n}_{\text{max}}\) but the solution coarsens with increasing time. Figure 6 shows that the exponential growth has again been curbed and that, in the algebraic regime, the perturbations produce a statistically steady \(\langle \partial_X v_{00} \rangle_V\) (Fig. 7). As required by our numerical approach, utilizing spatial averages only, we find that the rms fluctuations at saturation are small, 0.0035, while \(\langle (\partial_X v_{00})_V \rangle_t = 1.383\). Also included in this figure are the theoretical predictions (31) of the saturated state based on \(\hat{n}_{\text{max}}\) and that based on the smallest wavenumber permitted by our initial data. The latter case is in good agreement with the simulations.

### 4.2 Nonhomogeneous initial perturbations

Figure 8 illustrates the evolution of the poloidal stream and flux functions, \(\psi_{00}\) and \(\phi_{00}\), from an initial condition with random amplitudes of modes in both \(x\) and \(z\) in the computational domain used in Sect. 4.1.2. The initial state is composed of modes within \(\hat{p} \leq 2\sqrt{\chi}\) with a uniform sampling of amplitudes. As the flow evolves, all linearly stable spectral modes (satisfying \(\hat{p} > \sqrt{\chi}\)) are damped and the solution field is dominated by finger-like structures charac-
Fig. 3  The evolution of $\langle \partial X v_{00} \rangle_V$ for the Single Channel Mode solution in Fig. 2. Note that $\langle \partial X v_{00} \rangle_V = \partial X \Pi_{00}$ for the saturated state at late times.

Fig. 4  Evolution of $\phi_{00} / \langle \phi_{00} \rangle_{V}^{1/2}$ for a Multiple Channel Mode solution. For clarity we have only plotted the simulation until $t = 500$, and have separated the initial (left) and final (right) profiles. The profile coarsens to the largest seeded scale, $\tilde{n} = 2\pi/(8N_zL)$.

Fig. 5  The energy spectrum $E(|\tilde{p}|)$ at times $t = 5.5, 12, 50, 1000$ (solid, dotted, dotted, solid) for the Multiple Channel Mode solution shown in Fig. 4. The hash marks denote, from left to right, $\tilde{n}_{\text{box}}, \tilde{n}_{\text{max}}, \text{and } \tilde{n}_{\text{cutoff}}$. The energy of the modes near $n_{\text{max}}$ dominates early time solution, but the energy in these modes eventually saturates and the dominant energy moves upscale to lower modes.

4.3 Dissipation and saturation

We have seen that when $\Lambda \sim O \left( \frac{4}{3} \right) \gg 1$ the kinematic regime characterized by exponential growth gives way with increasing time to a regime of persistent and weaker algebraic growth. The explicit inclusion of subdominant dissi-
5 Discussion

In this section, we analyze the numerical results obtained in the previous section. Analytical progress can be made in the case of fully nonlinear single-mode solutions of the form (26). We discuss in turn the regimes of exponential growth, algebraic growth, and saturation when subdominant dissipation is included, and provide a detailed understanding of the numerical results.

5.1 Exponential growth regime

The initial kinematic regime is characterized by exponential growth that is captured in detail by neglecting the nonlinear terms $\partial Xv_{00}$, $\partial Xb_{00}$ in Eqs. (17)–(20). In this regime the dissipation terms of type $\epsilon_\nu \nabla^2$ or $\epsilon_\eta \nabla^2$ in Eqs. (17)–(20), where $\epsilon_\nu, \epsilon_\eta = \mathcal{O}(\Lambda^{-1})$ cuts off this slow growth and results in the complete saturation of the instability. For nonhomogeneous initial conditions the description of this saturation process requires, in addition, the retention of $\mathcal{O}((\epsilon_\delta \nabla^2)^{1/2})$ Jacobian nonlinearities (see Eqs. (17)–(20)). Figure 12 shows the evolution of a single-mode initial condition with $\hat{n} = \hat{n}_{\text{max}}$ in the case $\epsilon_\eta = 0.01, \epsilon_\nu = 0$ but keeping $\hat{\nu} = 1$. This choice of coefficients implies that viscous effects on fluctuating quantities are neglected in comparison to ohmic dissipation while viscous effects on mean quantities are retained (since the corresponding ohmic contribution $\langle b_{00} \rangle_V$ vanishes, see below). The fact that complete saturation is nonetheless observed indicates that the saturation process occurs primarily through the Reynolds and Maxwell stress terms, and not through viscous damping of the fluctuating fields. In contrast, when $\Lambda = \mathcal{O}(1)$ both ohmic dissipation and viscosity play a similar role in the saturation process (Knobloch & Julien 2005; Julien & Knobloch 2006; Julien & Knobloch 2007; Jamroz et al. 2008). Also included in Fig. 12 is the theoretical prediction of the final saturated state obtained in Sect. 5.2. Excellent agreement is found when the numerically determined dominant wavenumber $\hat{\nu}$ is employed in evaluating the theoretical expression.
contour plots of \( \frac{\psi_0}{\sigma_0} \) and \( \frac{\phi_0}{\sigma_0} \) for random \( x \)-dependent initial conditions at times \( t = 0 \) (NW), \( t = 10 \) (NE), \( t = 350 \) (SW) and \( t = 1000 \) (SE). The length \( L \) corresponds to the wavelength of the fastest growing wavenumber.

Fig. 9 Evolution of the rms fields in an \( x \)-dependent Multiple Mode solution in a log-log plot.

Fig. 10 Evolution of \( \langle \partial_x v_0 \rangle / \sqrt{V} \) for random \( x \)-dependent initial conditions. The quantities \( \partial_x \psi_{00 \text{single}} \) (dotted) and \( \partial_x \psi_{00 \text{eff}} \) (dashed), the theoretically predicted saturated values of \( \partial_x \psi_{00} \) from (31) based on \( \bar{b}_{\text{max}} \) and the smallest wavenumber \( \bar{b}_{\text{eff}} \) permitted, are included for reference.

instability evolves according to

\[
\begin{align*}
\langle \Psi_0(t), V_0(t), \Phi_0(t), B_0(t) \rangle \\
= (1, -\frac{\bar{n}_{\text{max}}}{\sigma}, -\bar{n}_{\text{max}}, 2\bar{n}_{\text{max}}^2) e^{\lambda_{\text{max}} t}.
\end{align*}
\]

This solution is in fact identical to the fully nonlinear, unarrested channel solution obtained by Goodman & Xu (1994). It maximizes the velocity and magnetic correlations in the shear stress tensor \( T_{xy} < 0 \) (25), corresponding to an out-
ward transport of angular momentum. However, Goodman & Xu implicitly omitted spatial derivatives on the large horizontal scale \( X \), and with them the dominant saturation mechanism when \( \Lambda \gg 1 \). The two-scale description given here shows that, as the perturbations grow, large-scale nonlinear terms become non-negligible due to the action of the MRI shear stress on the imposed background shear flow, and once this is the case the system can no longer be approximated by the linear system.

In Fig. 13 we show that the exponential growth of \( \langle \partial_X \nu_{00} \rangle \) computed from (23) tracks the predicted linear solution for early times. The time to the transition to algebraic growth can be determined from the value of \( \partial_X \nu_{00} \) determined in the next subsection.

5.2 Algebraic growth and the saturation of angular momentum transport

The numerical simulations of the Single Channel Mode case using Eqs. (17)–(20) show that the initial exponential growth gives way to algebraic growth/decay as time increases. Although the individual fields \( \psi_{00}, v_{11}, \phi_{00}, b_{11} \) grow or decay algebraically the quantity \( \partial_X \nu_{00} \) saturates at a constant value with small superposed oscillations (Fig. 2).

This observation suggests the Ansatz

\[
\begin{align*}
\psi_{00} &= (\Psi_1 t^{-\alpha} + \Psi_2 \cos \omega t) \cos(\hat{n}z) \\
v_{11} &= (V_1 t^\beta + V_2 \sin \omega t) \sin(\hat{n}z) \\
\phi_{00} &= (\Phi_1 t^\beta + \Phi_2 \sin \omega t) \sin(\hat{n}z) \\
b_{11} &= (B_1 t^{-\beta} + B_2 \cos \omega t) \cos(\hat{n}z),
\end{align*}
\]

where \( \alpha > 0, \beta > 0 \). Substitution into the reduced system (17)–(20) and (23)–(24) requires \( \alpha = \beta = \frac{1}{2} \) and yields, at

Fig. 11 Energy spectrum \( E(|\vec{p}|) \) at times \( t = 15, 50, 100, 1000 \) for random \( x \)-dependent initial conditions. The hash marks denote, from left to right, \( \hat{n}_{\text{box}}, \hat{n}_{\text{max}}, \) and \( \hat{n}_{\text{cutoff}} \).

Fig. 12 Nonlinear evolution of the MRI with the subdominant dissipative term \( \epsilon_\eta \nabla^2 \) explicitly included (with \( \epsilon_\eta = 0.01 \)). Parameter values are \( v_A = 1, \bar{v} = 1, \bar{\eta} = 1, \Omega = 1, \bar{\sigma} = -3\bar{\Omega}/2 \) with \( \bar{n} = \bar{n}_{\text{max}} \). The analytically predicted amplitudes of the saturated variables obtained in Sect. 5.3 are included for reference. Saturation is reached on a \( t_{\text{sat}} \sim O(\epsilon_\eta^{-1}) \) time scale.

Fig. 13 With no back-reaction the quantity \( \langle \partial_X \nu_{00} \rangle \) grows exponentially for all time, and rapidly exceeds the saturated value of \( \partial_X \nu_{00} \) predicted from the reduced Eqs. (17)–(20) with back-reaction via Eq. (23) included. Parameters are \( v_A = 1, \bar{v} = 1, \bar{\eta} = 1, \Omega = 1, \bar{\sigma} = -3\bar{\Omega}/2 \) with \( \bar{n} \) chosen to provide a modest growth rate \( \lambda = 0.1 \). The intersection of the exponentially growing solution and the saturated value of \( \partial_X \nu_{00} \) yields a lower bound on the time when the system is in a regime of algebraic growth/decay.
leading order,

\[
\begin{aligned}
(\Psi_1, V_1, \Phi_1, B_1) &= \left(1, \hat{\eta}^2/\Omega, -2\hat{\eta}, -2\hat{T}/v_A^2\right) \Psi_1, \\
(\Psi_2, V_2, \Phi_2, B_2) &= \left(1, -2\hat{\eta}^2/|\omega|, \hat{\eta}/|\omega|, \hat{\eta}^2/2|\Omega|\right) \Psi_2,
\end{aligned}
\]

(29)

(30)

together with the necessary conditions

\[
\partial_X \tau_{90} = -\hat{\sigma} - \frac{v_A^2\hat{\eta}^2}{2|\Omega|}, \quad \partial_X b_{90} = 0,
\]

(31)

and

\[
\omega = \sqrt{4\hat{\Omega}^2 + \hat{\eta}^2 v_A^2}.
\]

(32)

Equation (32) is the rms mean of the local inertial and Alfvén frequencies. Substituting the general form of the solutions (28) into the turbulent stress balance (Eq. (23) with volume averaging) and using the form of the eigensolutions (29, 30) yields

\[
\bar{\nu} (\partial_X v_{90})_V = \frac{\hat{\eta}^2 v_A^2}{2|\Omega|} \Psi_1^2 - \frac{\hat{\eta}^2}{8|\Omega|} \sin 2\omega t \Psi_2^2.
\]

(33)

It is remarkable that this expression does not contain secular terms proportional to \(t^{1/2} \cos \omega t, t^{-1/2} \sin \omega t\), let alone terms that grow in proportion to \(t\), and hence saturates despite the algebraic growth of the contributing fields. The mean component arises from products of the terms \(\Phi_1 t, V_1 t\) and \(\Psi_1 t^{-2}, B_1 t^{-2}\), while the oscillatory component is a consequence of the terms \((\Psi_2, V_2, \Phi_2, B_2)\). On time-averaging this result and comparing with Eq. (31) we obtain finally

\[
\Psi_1^2 = \frac{2\hat{\lambda}^2}{\hat{\eta}^2 v_A^2} \partial_X \tau_{90} = \frac{\bar{\nu} v_A^2}{\hat{\eta}^2 v_A^2} \left(\hat{\chi} - \hat{\eta}^2\right).
\]

(34)

We can numerically verify the relation (32) by measuring \(\omega\) for a range of values of \(\Omega\), with the remaining parameters fixed. Figure 14 shows perfect agreement with the analytical expression.

5.3 Dissipation and saturation

The weak (algebraic) growth of the MRI mode studied in the preceding section is easily overcome by small (subdominant) dissipation and results in true saturation of the instability on a dissipative time scale. Figure 12 shows the evolution of \(\psi_{90}, v_{11}, \Phi_{90}\) and \(b_{11}\) in the presence of the terms

\[
\epsilon_{\nu} \nabla^2, \epsilon_{\eta} \nabla^2
\]

with \(\epsilon_{\nu}, \epsilon_{\eta} = O(\hat{\nu}) \sim O(\Lambda^{-1})\). Although these terms do not affect the kinematic regime their presence is clearly felt at long times as the algebraic growth is curbed and the solution reaches an equilibrium state (Fig. 12). The resulting saturation amplitude can be computed following the approach of Knobloch & Julien (2005) and Julien & Knobloch (2006). We find

\[
\begin{aligned}
\Psi_{\text{sat}}, V_{\text{sat}}, \Phi_{\text{sat}}, B_{\text{sat}}(\tilde{\Omega}, \epsilon) = & \left(1, \frac{\hat{\nu}^2}{2|\Omega|}, \frac{1}{\epsilon_{\nu}}, \frac{1}{\epsilon_{\eta}}, \frac{(\epsilon_{\nu}-\epsilon_{\eta})\hat{\eta}^2}{2|\Omega|} - 4\hat{\Omega}^2 \epsilon_{\eta}\right) |\Psi_1|,
\end{aligned}
\]

(35)

where the non-homogeneous Multiple Modes, we obtain a regime of exponential growth followed by a transition to algebraic growth. As in the single mode case we see that the value of \(\partial_X v_{90}(\tilde{\Omega})\) approaches a statistically steady value in the algebraic growth regime (Figs. 7 and 10). However, the power spectra (Figs. 5 and 11) indicate that energy grows fastest in the smallest wavenumbers. Defining

\[
\partial_X \tau_{\text{obsingle}} = -\hat{\sigma} - \frac{v_A^2 \hat{\eta}^2}{2|\Omega|}
\]

and

\[
\partial_X \tau_{\text{eff}} = -\hat{\sigma} - \frac{v_A^2 \hat{\eta}^2}{2|\Omega|},
\]

Fig. 14 The frequency \(\omega\) of oscillations in the solution (28) as a function of \(\Omega\) with all other parameters are held constant: \(v_A = 1, \hat{\nu} = 1, \hat{\eta} = 1, \Omega = 1, \hat{\sigma} = -3\hat{\Omega}/2, \hat{\nu}\) corresponding to the fastest growing mode. The solid line represents the analytical relationship (32) while the diamonds indicate measured frequencies.

Note that the presence of both ohmic dissipation, \(\epsilon_{\eta} \neq 0\), and viscosity, \(\hat{\nu} \neq 0\), is required for saturation at finite but nonzero amplitude. Specifically, for \(\epsilon_{\eta} \gg \epsilon_{\nu}\) ohmic dissipation dominates and we find that \(\Psi_{\text{sat}}, V_{\text{sat}}, \Phi_{\text{sat}}, B_{\text{sat}}(\tilde{\Omega}) \sim O\left(\epsilon_{\eta}^{-1/2}, \epsilon_{\eta}^{-1/2}, \epsilon_{\eta}^{-1/2}, \epsilon_{\eta}^{-1/2}\right)\). Figure 12 shows that this theoretical prediction matches the numerical results and is achieved on the time scale \(t_{\text{sat}} \sim O\left(\epsilon_{\eta}^{-1}\right) \sim O(\Lambda^{-1})\). In contrast, when \(\epsilon_{\nu} \gg \epsilon_{\eta}\) dissipation is dominated by viscosity and the saturated state \((\Psi_{\text{sat}}, V_{\text{sat}}, \Phi_{\text{sat}}, B_{\text{sat}}(\tilde{\Omega}) \sim O\left(\epsilon_{\nu}^{-1/2}, \epsilon_{\nu}^{-1/2}, \epsilon_{\nu}^{-1/2}, \epsilon_{\nu}^{-1/2}\right)\) is reached on the time scale \(t_{\text{sat}} \sim O\left(\epsilon_{\nu}^{-1}\right)\).

5.4 Multiple modes

Using the preceding analysis for the Single Channel Mode, we can now interpret the results obtained for more general initial conditions. For both Multiple Channel Modes and the non-homogeneous Multiple Modes, we obtain a regime of exponential growth followed by a transition to algebraic growth. As in the single mode case we see that the value of \(\partial_X v_{90}(\tilde{\Omega})\) approaches a statistically steady value in the algebraic growth regime (Figs. 7 and 10). However, the power spectra (Figs. 5 and 11) indicate that energy grows fastest in the smallest wavenumbers. Defining

\[
\begin{aligned}
\partial_X \tau_{\text{obsingle}} = -\hat{\sigma} - \frac{v_A^2 \hat{\eta}^2}{2|\Omega|}
\end{aligned}
\]

and

\[
\begin{aligned}
\partial_X \tau_{\text{eff}} = -\hat{\sigma} - \frac{v_A^2 \hat{\eta}^2}{2|\Omega|},
\end{aligned}
\]

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where $\hat{p}_{\text{eff}}$ is the smallest wavenumber seeded by the initial condition, we can compare $\langle \partial_x v_{00} \rangle_Y$ with the saturated value of $\partial_x \pi_{00}$ predicted in Eq. (31) using single mode theory. We see that, although the mode corresponding to $\delta_{\text{max}}$ has the largest growth rate and dominates the kinematic phase, it is the mode with wavenumber $\hat{p}_{\text{eff}}$ that dominates the long time behavior. Figures 7 and 10 show that the quantity $\partial_x \pi_{\text{00eff}}$, computed using the numerically determined effective wavenumber, is a good predictor of the saturated value of $\langle \partial_x v_{00} \rangle_Y$.

6 Conclusion

In this article, we have extended the reduced model of the magnetorotational instability derived earlier (Knobloch & Julien 2005; Julien & Knobloch 2006; Julien & Knobloch 2007, Jamroz et al. 2008) to include two spatial scales in the horizontal, a fast scale $x$ comparable to the MRI scale in the vertical and a slow scale $X \equiv \delta x$. The former scale is necessary to resolve any boundary layers that may be present in the horizontal, while the latter captures the scale of the back-reaction of the MRI on the background shear. We have focused on the case of large Elsasser numbers, $\Lambda \gg 1$, appropriate for astrophysical accretion disks where the shear is the dominant source of energy for the instability, but the instability itself requires the presence of a vertical magnetic field. Dissipative processes, both ohmic and viscous, were taken to be subdominant as in prevailing accretion disk models, but not negligible. Although this model shares a linear dispersion relation with the ideal MHD system, the subdominant dissipative processes provide a mechanism whereby the local dynamics act on and modify the imposed background shear $\sigma$ through the generation of a turbulent shear stress $\tau_{xy} < 0$ that is perfectly balanced by a compensating viscous shear stress $\tau_{yz} > 0$. Thus the MRI evolves at the expense of reducing the imposed shear. Current shearing sheet approximations incorporate only the imposed background shear, $\Omega^* (r^*)$, but do not determine this shear self-consistently as the MRI grows in amplitude, i.e., current versions of the theory ignore the fact that the MRI in the nonlinear regime necessarily feeds off an evolving background. Recent global scale simulations by Cattaneo & Ombakko (private communication) provide clear evidence for the existence and importance of this process. We conjecture, following Julien & Knobloch (2006), that in the absence of horizontal boundaries the resulting shear profile may be stepwise constant on the scale $X$, with regions of MRI-reduced shear separated by shear zones on the scale $x$. Our computational domain is insufficient, however, to confirm this suggestion.

In this article we have demonstrated the importance of an evolving background shear for the saturation of the MRI. Numerical simulations of the extended reduced equations exhibit an initial kinematic phase of exponential growth familiar from current formulations of the instability process, followed by a transition to algebraic growth. Despite the continuing (algebraic) growth of some of the modes, the simulations indicate that the turbulent shear stress or equivalently the angular momentum transport saturates. The saturation is a consequence of the changing background shear and is the result of a balance between the algebraic growth of the azimuthal velocity and poloidal magnetic field fluctuations and the algebraic decay of the poloidal velocity and toroidal magnetic field fluctuations, resulting in saturated Reynolds and Maxwell stresses. Despite the absence of a traditional cascade in wavenumber the energy power spectrum reveals upscale transport to small wavenumbers as time increases, provided only that the instability is initiated with a broad spectrum of modes. When the subdominant dissipation terms are retained the algebraically growing MRI mode itself eventually saturates, and does so on a $\mathcal{O} (\delta / \epsilon) \sim \Lambda \gg 1$ time scale determined by ohmic (and not viscous) dissipation. The duration of the algebraic phase of the growth of the MRI is thus determined by $\Lambda$.

Our formulation of the MRI employs a negative shear $\sigma < 0$. The sign implicitly defines the ‘outward’ direction, and our result $\partial_x \pi_{00} > 0$ implies that $\tau_{xy} < 0$ always. Thus, in our reduced model the MRI always acts to transfer angular momentum outwards. It is of interest to obtain a relation estimating the $\alpha$ parameter of Shukla & Sunyaev (1973). Defining $W_{*r \theta} \equiv \alpha \epsilon^* \Delta^*$, where $\epsilon^* = \Omega^* H^*$ is the local sound speed and $W_{*r \theta}$ is the $r, \theta$ component of the local stress tensor $(W_{*r \theta} \equiv \langle u^* u^* \rangle_Y - \frac{1}{\rho \rho'} \langle a^* b^* \rangle_Y )$, in our asymptotic regime

$$\alpha = -\epsilon \delta \left( \frac{L^*}{H^*} \right)^2 \frac{\tau_{xy}}{\Omega^2} > 0,$$

where $L^* / H^* \ll 1$ denotes the characteristic MRI length scale determined in terms of the vertical pressure scale height.

We remark, finally, that the starting point for our investigation of the MRI was provided by the axisymmetric incompressible MHD equations in a cylindrical geometry. Allowing for non-axisymmetric dynamics offers potential for saturation of the MRI without the inclusion of sub-dominant dissipation as well as for the generation of the magnetic field by a dynamo process (Fleming et al. 2000; Rincon et al. 2007). In a future publication we will show that the reduced equations analyzed here, and their non-axisymmetric counterparts, can be rigorously deduced from the full compressible MHD equations (Umurhan & Regev 2004).

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