Strongly nonlinear magnetoconvection in three dimensions

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Abstract

Fully nonlinear three-dimensional convection in a strong vertical magnetic field is studied. In this regime, the convective velocities are not strong enough to distort the magnetic field substantially and the field remains primarily vertical. Consequently, the leading order nonlinearity arises from the distortion of the horizontally averaged temperature profile only. As a result all steady spatially periodic patterns have the same Nusselt numbers and mean temperature profile. A similar degeneracy is present in overstable convection with all periodic patterns having identical time-averaged Nusselt numbers and oscillation frequencies. These results are obtained via an asymptotic expansion in inverse Chandrasekhar number that determines, for each Rayleigh number, the time-averaged Nusselt number and oscillation frequency from the solution of a nonlinear eigenvalue problem for the vertical temperature profile. In the presence of variable magnetic Prandtl number \( \zeta \), these profiles are asymmetric, but nonetheless develop isothermal cores in the highly supercritical regime. The interesting case in which \( \zeta > 1 \) near the bottom (favoring steady convection) and \( \zeta < 1 \) near the top (favoring overstable convection) is discussed in detail.

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1. Introduction

The study of convection in an imposed magnetic field is motivated primarily by astrophysical applications, particularly by the observed magnetic field dynamics in the solar convection zone [1]. Applications to sunspots [2] have led several authors to investigate the suppression of convection by strong magnetic fields. The linear theory describing this suppression is summarized by Chandrasekhar [3]. Veronis [4] pointed out that nonlinear magnetoconvection can be subcritical, depending on the ratio \( \zeta \) of magnetic to thermal diffusivity, and consequently that convection could be present even when no linear instability is predicted. Subsequent studies of magnetoconvection [5] have targeted the two-dimensional problem, but even here the large magnetic field limit is inaccessible because of the stiffness of the resulting equations. The problem is even more acute in three dimensions, the study of which is only just beginning [6].

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In this paper we present a number of new solutions to the magnetoconvection problem. The solutions are fully nonlinear in the sense that they are valid at arbitrarily high Rayleigh numbers above onset, and moreover three-dimensional. They share one common characteristic in that they are all spatially periodic in the plane. These solutions are constructed via an asymptotic expansion in inverse powers of the Chandrasekhar number $Q$. This dimensionless number measures the strength of the imposed magnetic field and in the large $Q$ limit leads to a simplified set of dynamical equations. In these equations the dominant nonlinearity arises from the nonlinear distortion of the mean temperature profile; the strong magnetic field resists distortion by the velocity field and the Lorentz force arising from the distortion of the magnetic field remains small. As a result our solutions are characterized by a single wavenumber in the horizontal, with the vertical structure given by the solution of a nonlinear eigenvalue problem for the Nusselt number. The derivation of this nonlinear eigenvalue problem can be performed analytically, although the problem itself must be solved numerically, and can be performed for both steady and oscillatory magnetoconvection. We find that in the strong magnetic field limit all the competing steady patterns are degenerate in the sense that they transport the same amount of heat. This is so for the oscillatory patterns also. Although these results are formally obtained for stress-free boundary conditions, in the strong magnetic field limit they describe the dynamics outside of narrow boundary layers for other types of boundary conditions as well.

Of particular interest is the fact that our approach applies equally to the case in which the magnetic Prandtl number $\zeta$ (inverse Roberts number) depends nontrivially on the depth $d$ within the layer. Not only does this allow us to explore more realistic profiles of the magnetic and thermal diffusivities, but it also makes accessible the interesting case in which $\zeta$ passes through one somewhere in the layer. According to linear theory [3], $\zeta > 1$ favors steady convection while $\zeta < 1$ favors oscillatory convection. We show here that these conclusions extend to the strongly nonlinear regime even when $\zeta$ depends on $z$. However, near the surface of the solar convection zone ($1500 \text{ km} < d < 20000 \text{ km}$), the thermal diffusivity is reduced owing to the increase in opacity caused by ionization and in this region $\zeta > 1$. Both above and below this region, $\zeta < 1$. It has been suggested [7,8] that these changes in $\zeta$ are responsible for the presence of umbral dots with the more efficient steady convection penetrating into the overlying regions of less efficient overstable convection and forming the intermittent bright spots observed in sunspot umbrae. Our solutions in this regime lend further support to this idea. We find strongly nonlinear two- and three-dimensional solutions in which overturning convection in the lower part of the layer is coupled to overstable convection in the upper part. The resulting solution is periodic in time but the oscillation amplitude is small near the bottom and large near the top. Moreover, the oscillation period becomes independent of the applied Rayleigh number at high Rayleigh numbers indicating that the oscillation is of magnetic origin. We do not consider compressibility effects.

This paper is organized as follows. In the next section we introduce the governing equations and summarize their linear and weakly nonlinear properties. In Section 3 we describe the asymptotic expansion that leads to a new class of reduced equations describing fully nonlinear convection in the strong magnetic field limit. These equations take the form of coupled equations for the streamfunction and fluxfunction, driven by thermal buoyancy. The equations for vortical velocity and associated magnetic field decouple and these components decay away. The resulting equations form a simplified set of equations governing three-dimensional convection in a strong magnetic field at arbitrary Rayleigh numbers. Although these equations are of interest in their own right, we focus here on spatially periodic solutions only. In Section 4 we describe steady and oscillatory solutions with different planforms, and these are compared with the corresponding results for variable $\zeta$ in Section 5. In Section 6 we compare the results of an analytical solution of the nonlinear eigenvalue problem for asymptotically high Rayleigh numbers and constant $\zeta$ with numerical solutions. The paper ends with a discussion of the implications of the fully nonlinear three-dimensional solutions constructed here for ongoing simulations of magnetoconvection.
2. Governing equations

The dimensionless Boussinesq equations describing magnetoconvection in a plane horizontal layer of height \( h \) are

\[
\frac{1}{\sigma} \frac{D \mathbf{u}}{Dt} = -\nabla \pi + \zeta Q \mathbf{B} \cdot \nabla \mathbf{B} + \text{Ra} \nabla^2 \mathbf{u},
\]

\[
\frac{DT}{Dt} = \nabla^2 T,
\]

\[
\frac{DB}{Dt} = \mathbf{B} \cdot \nabla \mathbf{u} - \nabla \times (\zeta \nabla \times \mathbf{B}),
\]

\[
\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0,
\]

where \( \mathbf{u} = (u, v, w) \) is the velocity field in Cartesian coordinates \((x, y, z)\) with \( z \) vertically upwards. We use the symbols \( T \) to denote the temperature and \( \pi \) the total (thermal and magnetic) pressure. The dimensionless magnetic field is assumed to be the superposition \( \mathbf{B} = \hat{z} + \mathbf{b} \) of an imposed vertical field of unit strength in the \( z \)-direction and a three-dimensional field \( \mathbf{b} \) due to the presence of convection. The equations have been nondimensionalized with respect to the thermal diffusion time in the vertical. The resulting dimensionless parameters

\[
Q = \frac{B_0^2 h^2}{\mu_0 \rho \eta v}, \quad \text{Ra} = \frac{g \alpha \Delta T h^3}{v \kappa}, \quad \sigma = \frac{v}{\kappa}, \quad \zeta = \frac{\eta}{\kappa},
\]

are the Chandrasekhar, Rayleigh, and thermal and magnetic Prandtl numbers, respectively. Here we consider the case where \( \sigma \) is a constant, but \( \zeta \equiv \zeta(z) \) is allowed to vary with height.

We employ a streamfunction formulation and write

\[
\mathbf{u}(x, y, z, t) = \nabla \times \phi(x, y, z, t) \hat{z} + \nabla \times \nabla \times \psi(x, y, z, t) \hat{z},
\]

\[
\mathbf{b}(x, y, z, t) = \nabla \times A(x, y, z, t) \hat{z} + \nabla \times \nabla \times B(x, y, z, t) \hat{z}.
\]

Thus,

\[
\mathbf{u} = \begin{pmatrix} \partial_x \phi + \partial_y \partial_z \psi \\ -\partial_z \phi + \partial_y \partial_z \psi \\ -\nabla^2 \psi \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \partial_x A + \partial_y \partial_z B \\ -\partial_z A + \partial_y \partial_z B \\ -\nabla^2 B \end{pmatrix},
\]

\[
\omega = \begin{pmatrix} \partial_x \partial_y \phi - \nabla^2 \partial_z \psi \\ \partial_y \partial_x \phi + \nabla^2 \partial_z \psi \\ -\nabla^2 \partial_x \phi \end{pmatrix}, \quad j = \begin{pmatrix} \partial_x \partial_y A - \nabla^2 \partial_z B \\ \partial_y \partial_x A + \nabla^2 \partial_z B \\ -\nabla^2 A \end{pmatrix},
\]

where \( \omega \equiv \nabla \times \mathbf{u} \) and \( j \equiv \nabla \times \mathbf{b} \) are, respectively, the vorticity and the current density. Partialis with subscripts denote differentiation with respect to that variable, \( \partial_z \equiv \partial/\partial z \), and \( \nabla^2 \equiv \partial_{xx} + \partial_{yy} \) is the horizontal Laplacian.

It is a simple matter now to write down equations for the four scalar fields \( \phi, \psi, A, \) and \( B \):

\[
\frac{1}{\sigma} (\partial_t \nabla^2 \phi + N \phi) = \zeta Q (D \nabla^2 A + M \phi) + \nabla^2 \nabla^2 \phi,
\]

\[
\frac{1}{\sigma} (\partial_t \nabla^2 \psi + N \psi) = -\text{Ra} \nabla^2 T + \zeta Q (D \nabla^2 \nabla^2 B + M \psi) + \nabla^4 \nabla^2 \psi,
\]

\[
\partial_t \nabla^2 A + M_A = D \nabla^2 \phi + \zeta \nabla^2 \nabla^2 A + (D \xi) \nabla^2 A,
\]

\[
\partial_t \nabla^2 B + M_B = D \nabla^2 \psi + \zeta \nabla^2 \nabla^2 B,
\]
where $D \equiv \partial / \partial z$. The definitions of the nonlinear advection terms $N_\phi, M_\phi, N_\psi, M_\psi, M_A$ and $M_B$ have been relegated to the appendix.

These equations are solved for a fluid confined between boundaries at fixed temperatures,

$$T(0) = 1, \quad T(1) = 0,$$

which are impenetrable and either stress-free or no-slip

\begin{align*}
\text{stress-free} : & \quad \psi = \partial_z \psi = \partial_z \phi = 0 \\
\text{no-slip} : & \quad \psi = \partial_z \psi = \phi = 0 \quad \text{at } z = 0, 1.
\end{align*}

(14)

The magnetic boundary conditions will be taken as

$$A = \partial_z A = \partial_z B = 0, \quad \text{at } z = 0, 1$$

(15)

although other boundary conditions can be considered as well. Throughout we use periodic boundary conditions in the horizontal; these are equivalent to formulating the equations on a periodic lattice [9] as discussed further below.

2.1. Linear theory

The stability properties of the conduction state

$$\psi = \phi = A = B = 0, \quad T = 1 - z$$

(16)

are summarized by Chandrasekhar [3]. In the strong magnetic field limit these results become independent of the nature of the velocity boundary conditions at the top and bottom of the layer [3]. In this limit one finds that for $\zeta > 1$ the conduction state loses stability at a steady state bifurcation at

$$Ra_c^{(s)} = \pi^2 Q, \quad k_c^{(s)} = \left[ \frac{1}{2} \pi^4 Q \right]^{1/6}.$$ (17)

while for $\zeta < 1$ it loses stability to overstable oscillations (Hopf bifurcation) at

$$Ra_c^{(o)} = \left[ \frac{\zeta (\sigma + \zeta)}{1 + \sigma} \right] \pi^2 Q,$$

(18)

with

$$k_c^{(o)} = \left[ \frac{1}{2} \pi^2 Q \frac{\sigma \zeta}{(1 + \sigma)(1 + \zeta)} \right]^{1/6}, \quad \omega_c^{(o)} = \left[ \frac{1 - \zeta}{1 + \sigma} \right]^{1/2} \pi (\sigma \zeta Q)^{1/2}.$$ (19)

Here $\omega_c^{(o)}$ is the Hopf frequency. Fig. 1 compares the exact solutions of the linear stability problem for $Q = 10^7$ with the asymptotic relations (17) and (18) when $\zeta = 0.1, \sigma = 1.0$.

The above scaling does not capture the transition from oscillatory onset to steady state onset. As shown by Chandrasekhar [3], this point is characterized by zero oscillation frequency and occurs when

$$Q = \frac{\zeta (1 + \sigma) (\pi^2 + k^2)^2}{\sigma (1 - \zeta) \pi^2},$$

(20)

where $k$ is the horizontal wavenumber. It follows that in order to capture this transition one must take $k = \mathcal{O}(Q^{1/4})$ for large $Q$. When this is the case, the corresponding Rayleigh number is given by

$$Ra = \frac{\sigma + \zeta}{\sigma (1 - \zeta)} \frac{(\pi^2 + k^2)^3}{k^2},$$

(21)

and remains $\mathcal{O}(Q)$ for large $Q$. In the following we refer to this transition point as the Takens–Bogdanov point.
2.2. Weakly nonlinear theory

Weakly nonlinear three-dimensional magnetoconvection at finite $Q$ was studied by Clune and Knobloch [10] using the stress-free boundary conditions (14) and (15) in the vertical and periodic boundary conditions in the horizontal. The problem was formulated on a periodic lattice. In the simplest case such a formulation selects four wavenumbers (square lattice) or six wavenumbers (hexagonal lattice) from the circle of marginally stable wavenumbers. A steady bifurcation in the former case describes the competition between rolls (R) and squares (S). Clune and Knobloch found that squares are always unstable with respect to rolls, and that rolls are stable with respect to perturbations on the square lattice if they are supercritical. In the latter case the resulting bifurcation provides a description of the competition between rolls (R), hexagons (H), regular triangles (RT) and a pattern of rectangles called patchwork quilt (PQ), but rolls remain the preferred pattern. The possibilities are much richer when the primary instability is oscillatory. On the square lattice five solutions bifurcate simultaneously from the conduction state: traveling rolls (TR), standing rolls (SR), standing squares (SS), traveling squares (TS) and alternating rolls (AR). On the hexagonal lattice there are at least eleven primary solution branches. In each case Clune and Knobloch use weakly nonlinear theory to identify the regions in the $(\sigma, Q)$ plane for several values of the parameter $\zeta < 1$ containing different stable solutions. These calculations indicate that for $Q = 10^{10}$ the alternating rolls are always stable near onset with respect to perturbations on the square lattice, and that for $\sigma > \sigma_c(\xi)$ these coexist with stable traveling rolls. The situation is simpler on the hexagonal lattice where the only stable pattern at $Q = 10^{10}$ are oscillating triangles (OT).

The stability predictions from the weakly nonlinear theory are expected to hold even for fully nonlinear convection. For such solutions we find that if the solution frequency $\omega$ remains bounded away from zero the branches exist globally. Moreover, if such a branch is stable near onset it can lose stability only at a secondary saddle-node bifurcation, at a secondary Hopf bifurcation, or at a parity-breaking bifurcation. If the amplitude increases monotonically with $Ra$, no saddle-node bifurcations are present. A secondary Hopf bifurcation from the TR branch generates a branch of two-frequency modulated waves which may terminate on one of the other branches in a parity-breaking bifurcation [11]. Hopf bifurcations from SR and SS may take the form of two-frequency standing waves or three-frequency standing waves that drift back and forth [12]. Parity-breaking bifurcations are steady state bifurcations from a group orbit of non-drifting solutions that result in a drift. Such drifting standing waves are two-frequency states and often connect with the two-frequency waves generated by secondary Hopf bifurcations from traveling states. None of these bifurcations can be found without an explicit stability calculation. Additional wavelength changing instabilities are also possible, but require the use of periodic boundary conditions based on multiple wavelengths of the basic state. We do not pursue these possibilities here.
3. Derivation of the reduced equations

Simplified equations can be obtained using the scaling

\[ \partial_x, \partial_y = Q^{1/4}(\partial_{x'}, \partial_{y'}), \quad \text{Ra} = Q \text{Ra}', \quad \partial_z = Q^{1/2} \partial_{z'}, \]  

where \( Q \gg 1 \). With this scaling we focus on small horizontal scales (and high frequency oscillations in the case of overstable convection). As discussed in Section 2, this scaling is the appropriate one for capturing the transition from steady to overstable convection within linear theory, i.e., the Takens–Bogdanov point. Although it is not the scaling that captures the modes that first go unstable as Ra increases (see Section 2), we show below that our results with this scaling do reduce to the correct results for \( \mathcal{O}(Q^{1/6}) \) wavenumbers in the appropriate limit. The scaling (22) therefore describes a much broader range of wavenumbers than apparent at first sight, and in particular retains the wavenumber dependence of the problem. In contrast to earlier work along these lines [13], the Prandtl numbers \( \sigma \) and \( \zeta \) are not scaled and remain of order one. The same scaling works for both stress-free and no-slip boundary conditions at the top and bottom. The fluid variables are scaled according to

\[ \phi = Q^{-1/4} \phi_1(x', y', z, t') + \mathcal{O}(Q^{-1/2}), \]  

\[ \psi = Q^{-1/4} \psi_1(x', y', z, t') + \mathcal{O}(Q^{-1/2}), \]  

\[ A = Q^{-3/4} A_1(x', y', z, t') + \mathcal{O}(Q^{-1}), \]  

\[ B = Q^{-3/4} B_1(x', y', z, t') + \mathcal{O}(Q^{-1}), \]  

\[ T = \theta_0(z) + Q^{-1/4} \theta_1(x', y', z, t') + Q^{-1/2} \theta_2(x', y', z, t') + \mathcal{O}(Q^{-3/4}), \]

and we write \( D = \partial_z \). With this scaling the horizontal components of the velocity \( \mathbf{u} \) and magnetic field perturbation \( \mathbf{b} \) are \( \mathcal{O}(1) \) and \( \mathcal{O}(Q^{-1/2}) \), respectively, while the vertical components are \( \mathcal{O}(Q^{1/4}) \) and \( \mathcal{O}(Q^{-1/4}) \). Thus, the convective velocities are substantial (specifically the convective kinetic energy \( E \) in a convection cell is \( \mathcal{O}(1) \)) but the magnetic field remains primarily vertical. It follows that

\[ M_\phi = \mathcal{O}(Q^{-1/2}), \quad N_\phi = \mathcal{O}(Q^{1/2}), \quad M_\psi = \mathcal{O}(1), \quad N_\psi = \mathcal{O}(Q), \]  

\[ M_A = \mathcal{O}(1), \quad M_B = \mathcal{O}(1), \]

and hence (dropping primes) that

\[ \frac{1}{\sigma} \partial_t \phi_1 = \zeta(z) D A_1 + \nabla_\perp^2 \phi_1 + \mathcal{O}(Q^{-1/4}), \]  

\[ \frac{1}{\sigma} \partial_t \nabla_\perp \psi_1 = -\text{Ra} \partial_t + \zeta(z) D \nabla_\perp^2 B_1 + \nabla_\perp^4 \psi_1 + \mathcal{O}(Q^{-1/4}), \]  

\[ \partial_t A_1 = D \phi_1 + \zeta(z) \nabla_\perp^2 A_1 + \mathcal{O}(Q^{-1/4}), \]

\[ \partial_t B_1 = D \psi_1 + \zeta(z) \nabla_\perp^2 B_1 + \mathcal{O}(Q^{-1/4}). \]

Note that Eqs. (29)–(32) are linear at leading order and that spatial derivatives of \( \zeta \) are absent. This is a consequence of the small horizontal scales considered. In the manipulations that follow it is important, however, to retain the depth dependence of \( \zeta \) since these reintroduce such derivatives, and these can have a large effect on the leading order solution.
The nonlinear effects enter into the problem only through the temperature equation which yields at $O(Q^{1/4})$
\[
\partial_t \theta_1 - \nabla_\perp^2 \theta_1 = \nabla_\perp^2 \psi_1 D \theta_0,
\] (33)
and at $O(1)$
\[
\partial_t \theta_2 - J(\phi_1, \psi_1) + \nabla_\perp D \psi_1 \cdot \nabla_\perp \theta_1 - \nabla_\perp^2 \psi_1 D \theta_1 - \nabla_\perp^2 \psi_2 D \theta_0 = \nabla_\perp^2 \theta_2 + D^2 \theta_0.
\] (34)
The solvability condition for the mean part of $\theta_2$ now yields
\[
D^2 \theta_0 + D(\nabla_\perp^2 \psi_1 \theta_1) = 0
\] (35)
which can be integrated once, obtaining
\[
D \theta_0 + \nabla_\perp^2 \psi_1 \theta_1 = -K.
\] (36)
For steady patterns the constant $K$ is identified with the Nusselt number; for oscillatory patterns we extend the meaning of the overbar to indicate an average over time as well. For such patterns $K$ represents the time-averaged Nusselt number. Eqs. (33) and (35), together with Eqs. (30) and (32) form a closed system of equations; the equations for $\phi_1$ and $A_1$ decouple. In the absence of forcing these variables both decay to zero (see below).

It is worth mentioning that because the vortical fields $\phi$ and $A$ decouple, they can be scaled independently of the other fields. For example, if we start with larger vortical perturbations, we can write
\[
\phi = \phi_1(x, y, z, t) + O(Q^{-1/4}), \quad A = Q^{-1/2}A_1(x, y, z, t) + O(Q^{-3/4}).
\] (37)
This Ansatz leads to the following evolution equations
\[
\frac{1}{\sigma}[\partial_t \nabla_\perp^2 \phi_1 - J(\phi_1, \nabla_\perp^2 \phi_1)] = \zeta D A_1 - \zeta J(A_1, \nabla_\perp^2 A_1) + \nabla_\perp^4 \phi_1 + O(Q^{-1/4}),
\] (38)
\[
\frac{1}{\sigma}[\partial_t \nabla_\perp^2 \psi_1 - J(\phi_1, \nabla_\perp^2 \psi_1)] = -Ra \theta_1 + \zeta D \nabla_\perp^2 B_1 - \zeta J(A_1, \nabla_\perp^2 B_1) + \nabla_\perp^4 \psi_1 + O(Q^{-1/4}),
\] (39)
\[
\partial_t A_1 - J(\phi_1, A_1) = D \phi_1 + \zeta \nabla_\perp^2 A_1 + O(Q^{-1/4}),
\] (40)
\[
\partial_t B_1 - J(\phi_1, B_1) + J(A_1, \nabla_\perp^2 \psi_1) = D \nabla_\perp^2 \psi_1 + \zeta \nabla_\perp^4 B_1 + O(Q^{-1/4}).
\] (41)
The ($\phi_1, A_1$) equations therefore still decouple but with appropriate vorticity injection these equations can now exhibit highly nontrivial dynamics [14]. Note, however, that the nonlinear terms in the ($\psi_1, B_1, \theta_1$) equations vanish for all steady patterns in the form of R, S, H, RT and PQ, as well as for standing oscillatory patterns in the form of rolls, squares, hexagons, regular triangles and patchwork quilt, and for traveling rolls. Consequently, fully nonlinear solutions of these types can still be obtained and these are in fact identical to the solutions of the nonlinear eigenvalue problem derived below with the original scaling. However, some solutions, notably AR, TS and six other solutions on the hexagonal lattice no longer satisfy the equations. Such solutions can still be found via weakly nonlinear theory, but no longer at arbitrary supercriticality. This is because the necessary condition,
\[
\partial_x h \partial_x h^* - \partial_x h^* \partial_x h = 0,
\] (42)
for the horizontal Jacobians to vanish does not hold for these patterns. As a consequence such patterns no longer have a single wavenumber in the horizontal.

In the following we focus on fully nonlinear solutions of Eqs. (29)–(34) that are periodic in space and time. Because of the simple structure of these equations, such solutions have simple horizontal structure for all values of Ra. In the next section we describe patterns of this type.
4. Steady and oscillatory patterns

In this section we use Eqs. (29)–(34) to construct a number of spatially periodic solutions at arbitrarily supercritical Rayleigh numbers. To this end we introduce planform functions \( h(x, y) \) satisfying the planform equation

\[
\nabla^2 h + \kappa^2 h = 0.
\]

(43)

These functions are real for steady patterns and standing oscillatory patterns. For these patterns, it is convenient to normalize \( h \) such that \( |h|^2 = 1 \). For oscillatory patterns of traveling type \( h \) is in general a complex-valued function and we normalize it so that \( |h|^2 = 1 \). The planforms of interest are solutions to this Helmholtz equation that are periodic in the plane and isotropic. These include rolls \( h = \sqrt{2} \cos kx \), squares \( h = \cos kx + \cos ky \), hexagons \( h = \sqrt{2/3} (\cos kx + \cos((1/2)kx + (\sqrt{3}/2)ky) + \cos((1/2)kx - (\sqrt{3}/2)ky)) \), regular triangles \( h = \sqrt{2/3} (\sin kx + \sin((1/2)kx + (\sqrt{3}/2)ky) + \sin((1/2)kx - (\sqrt{3}/2)ky)) \), and patchwork quilt \( h = \cos((1/2)kx + (\sqrt{3}/2)ky) + \cos((1/2)kx - (\sqrt{3}/2)ky) \). We use the terminology patchwork quilt to distinguish this special solution from general rectangles which also satisfy the planform equation \( h = \cos mx + \cos ny, m^2 + n^2 = k^2 \). Similar expressions apply in the overstable case.

In the following we therefore write

\[
(\psi_1, \phi_1, A_1, B_1, \theta_1) = \frac{1}{2} [\psi_1(z), \phi_1(z), A_1(z), B_1(z), \theta_1(z)] h(x,y) e^{i\omega t} + \text{c.c.}
\]

(44)

Then

\[
(i\omega + \zeta k^2)A_1 = D\phi_1, \quad (i\omega + \zeta k^2)B_1 = D\psi_1,
\]

\[
\left(\frac{i\omega}{\sigma} + k^2\right)\phi_1 = \zeta DA_1, \quad \left(\frac{i\omega}{\sigma} + k^2\right)k^2\psi_1 = Ra \theta_1 + \zeta k^2 DB_1.
\]

(45)

Moreover,

\[
(i\omega + k^2)\theta_1 = -k^2 \psi_1 D\theta_0,
\]

(46)

so that

\[
D\theta_0 \left[ 1 + \frac{k^6}{2(\omega^2 + k^4)} |\psi_1|^2 \right] = -K,
\]

(47)

where

\[
K^{-1} = \int_0^1 \frac{\omega^2 + k^4}{\omega^2 + k^4 + (1/2)k^6 |\psi_1|^2} \, dz.
\]

(48)

Combining the above expressions, we obtain the nonlinear eigenvalue problem

\[
D^2 \psi_1 - \frac{(\Delta \zeta)k^2}{i\omega + \zeta k^2} D\psi_1 - \frac{1}{\zeta} \left(\frac{i\omega}{\sigma} + k^2\right) (i\omega + \zeta k^2) \psi_1 + \frac{Ra K}{\zeta} \frac{(i\omega + \zeta k^2)(-i\omega + k^2)}{\omega^2 + k^4 + (1/2)k^6 |\psi_1|^2} \psi_1 = 0.
\]

(49)

The solutions of this problem depend on the prescribed function \( \zeta(z) \) as well as the parameters \( Ra, k, \) and \( \sigma \). In the case where \( \zeta \) is constant, Eq. (50) reduces to

\[
D^2 \psi_1 - \frac{1}{\zeta} \left(\frac{i\omega}{\sigma} + k^2\right) (i\omega + \zeta k^2) \psi_1 + \frac{Ra K}{\zeta} \frac{(i\omega + \zeta k^2)(-i\omega + k^2)}{\omega^2 + k^4 + (1/2)k^6 |\psi_1|^2} \psi_1 = 0.
\]

(50)

Note that steady solutions \( (\omega = 0) \) are independent of both \( \sigma \) and \( \zeta \). The latter is not generally true and for finite \( Q \) the steady solutions do depend on \( \zeta \).
In contrast, the $\phi_1$ and $A_1$ equations yield the linear eigenvalue problem
\[
D^2 \phi_1 - \frac{(D\zeta)k^2}{i\omega + \zeta k^2} D\phi_1 = \frac{1}{\zeta} \left( \frac{i\omega}{\alpha} + k^2 \right) (i\omega + \zeta k^2)\phi_1,
\]
which if $\zeta$ is constant becomes
\[
D^2 \phi_1 = \frac{1}{\zeta} \left( \frac{i\omega}{\alpha} + k^2 \right) (i\omega + \zeta k^2)\phi_1.
\]
Eqs. (50) and (51) are to be solved subject to the boundary conditions
\[
\psi_1(0) = \psi_1(1) = 0
\]
imposing impermeability of the boundaries. These boundary conditions are independent not only of the velocity boundary conditions at the top and bottom, but also of the details of the magnetic boundary conditions. Consequently, the solutions of the nonlinear eigenvalue problem describe the solutions in the bulk of the layer, outside of thin passive Hartmann boundary layers next to the boundaries [15]. Since both eigenvalue problems are invariant under $\psi_1 \rightarrow -\psi_1$ the direction of flow in any solution can be reversed, and consequently in the strong magnetic field limit there is no distinction between $\pm$ hexagons, even if $\zeta(z)$ is not symmetric with respect to midheight. These, like the remaining steady patterns, satisfy the same eigenvalue problem and consequently have the same Nusselt number $K$ and bifurcate in the same direction. Likewise, all oscillatory patterns with the same wavenumber $k$ have the same (time-averaged) Nusselt numbers $K$ and frequencies $\omega$. This degeneracy is a consequence of the fact that in the strong magnetic field limit the leading nonlinearity is provided by the horizontally averaged temperature profile and is of course present in the large $Q$ limit of the weakly nonlinear theory as well. Note that it is not necessary to calculate $\psi_2$. This term usually describes mean flows associated with traveling patterns, but in the strong field limit such flows are negligible, even for fully nonlinear states.

Before solving the resulting problem, we must choose the wavenumber $k$ of the instability. Usually this wavenumber is chosen to be that which minimizes the onset Rayleigh number. With the present scaling (see Eq. (22)) this minimum occurs at $k = 0$ (see Eqs. (17) and (19)); in this limit the nonlinear eigenvalue problem ((51),(54)) reduces to the results obtained for $O(Q^{1/6})$ wavenumbers by Matthews [16], and this is so for both steady and oscillatory convection. Thus the present formulation describes correctly not only the fully nonlinear solutions present near the transition from steady to oscillatory convection but also captures a much wider range of wavenumbers that includes the wavenumbers for steady and oscillatory convection selected at onset. We are therefore free to choose the wavenumber $k$ from a broad range and all our calculations use $k = 1$. We solve the eigenvalue problems ((50),(51)) on a discretised one-dimensional mesh using an iterative Newton–Raphson–Kantorovich (NRK) scheme [17,18] with $O(10^{-10})$ accuracy in the $L_2$ norm of $\psi_1(z)$ and comparable accuracy in the corresponding eigenvalues $K$ and $\omega$. Both steady ($\omega = 0$) and overstable ($\omega \neq 0$) solutions are constructed. We consider first the case of constant $\zeta$ described by Eq. (51). In Fig. 2, we show the (time-averaged) Nusselt number $K$ and frequency for both steady and overstable convection as a function of the scaled Rayleigh number $Ra$. Observe that solutions can be obtained for highly supercritical Rayleigh numbers and that $K$ increases monotonically with increasing $Ra$, while the frequency $\omega$ appears to saturate. Figs. 3 and 4 show the corresponding mean temperature profiles $\theta_0 \equiv \bar{T}(z)$ for several values of $Ra$. For both steady and oscillatory convection the temperature gradients are confined to thinner and thinner boundary layers at the top and bottom as $Ra$ increases. At the same time the bulk of the layer becomes more and more isothermal. Note that these boundary layers are symmetrical with respect to $z = 1/2$ and that the isothermal interior has temperature $T = 1/2$, i.e., a temperature that is exactly half way between the temperatures at the top and bottom boundaries. Fig. 5 shows the associated magnetic perturbation in one two-dimensional convection cell in
Fig. 2. (a) The (time-averaged) Nusselt number $K$ for a steady (solid line) and oscillatory (dashed line) convection as a function of the scaled Rayleigh number $Ra$ when $\zeta = 0.1$ and $\sigma = 1.1$. (b) The corresponding (scaled) oscillation frequency $\omega$. Note that $\omega$ appears to saturate with increasing $Ra$.

Fig. 3. Mean temperature profiles $\bar{T}(z)$ for steady convection at several values of the (scaled) Rayleigh number showing the development of an isothermal core with increasing $Ra$ when $\zeta = 0.1$.

terms of the contours of the perturbation potential $B_1$ computed from Eq. (45). Observe that these perturbations are confined to the vicinity of the top and bottom boundaries; no lateral flux expulsion occurs in this problem because the magnetic field remains harmonic in the horizontal. In terms of the unscaled variables these perturbations are $O(Q^{-1/4})$ and hence small, i.e., at leading order the total magnetic field remains vertical and uniform.
The solution of the nonlinear eigenvalue problem can now be used to construct solutions in the form of the different possible three-dimensional patterns described at the beginning of this section. Although we do not show such patterns here (see [19] for such reconstructions), we note that all standing patterns have vertical boundaries. This is not so for traveling planforms such as TW and TS [20]. Moreover, for these patterns the instantaneous Nusselt numbers are both time-independent and \( z \)-independent, as required by translation invariance [21]. In contrast, the instantaneous Nusselt numbers for SR and SS oscillate in time with frequency \( 2\omega \) and depend on \( z \). This is so also for the specific kinetic energy density \( \tilde{\mathcal{E}} \).

For example, for a standing square pattern

\[
\tilde{\mathcal{E}} = \pi^2 |D\psi_1|^2 + \pi^2 \kappa^2 |\psi_1|^2 + \frac{\pi^2}{2} \left\{ [(D\psi_1)^2 + k^2 \psi_1^2] e^{2i\omega t} + \text{c.c.} \right\} + \mathcal{O}(Q^{-1/4}).
\]  

In Fig. 6(a) we show \( \langle E \rangle \), the time-averaged specific kinetic energy \( E \equiv \int \tilde{\mathcal{E}} \, dz \), as a function of Ra for both steady and oscillatory square pattern convection; Fig. 6(b) shows \( E(t) \) for a standing square pattern at Ra = 11615. For the value of \( \zeta \) used (\( \zeta = 0.1 \)) oscillatory convection sets in first (Fig. 1) and hence has a larger amplitude than the corresponding steady state at each value of Ra. Note that the specific kinetic energy is strictly positive at all times (i.e., there is no instant in time at which the fluid is wholly at rest), and that the time-averaged specific kinetic energy increases monotonically with increasing Ra.

5. Results for variable \( \zeta \)

In this section we present the solutions of the nonlinear eigenvalue problem (50) with \( \zeta(z) = \zeta_0 + \epsilon(1 - z) \). We choose a linear variation of \( \zeta \) in order to be able to compare our results with simulations of magnetic fields in \( m = 1 \) polytropic atmospheres carried out by Hurlburt and Toomre [22] and Weiss et al. [8,23]. Since our calculations do not include any effects of compressibility all vertical inhomogeneities come from the variation of \( \zeta \). Consequently the results that follow allow us to discriminate between effects due to variation of \( \zeta \) and those due to compressibility, at least in the strong field regime.

In the following we set \( \zeta_0 = 1.0 \) or \( \zeta_0 = 0.2 \) and increase the parameter \( \epsilon \), i.e., the strength of the nonuniformity of \( \zeta \). In the former case \( \zeta > 1 \) everywhere in the cell and the convective instability is necessarily steady. In
Fig. 5. Contours of constant magnetic vector potential for the perturbation magnetic field $\mathbf{b}$ in a two-dimensional cell $|x| \leq \pi/k$, $0 \leq z \leq 1$ for (a) steady convection at $Ra = 8200$ and (b) oscillatory convection at six equally spaced times $t = 0, (\pi/3\omega), 2\pi/3\omega, 0, \pi/3, \pi/3$. The parameters are $\zeta = 0.1$ and $\sigma = 1.1$. Solid (dashed) lines indicate positive (negative) contributions.
Fig. 6. (a) The time-averaged kinetic energy $\langle E \rangle$ per unit mass for oscillatory (dashed line) and steady (solid line) convection with square planform as a function of the (scaled) Rayleigh number $Ra$ when $\xi = 0.1, \sigma = 1.1$. (b) The kinetic energy $E(t)$ for standing squares at $Ra = 11,615$.

the latter case $\xi$ varies with depth in such a way that it is larger at the bottom than at the top, and we choose values of $\epsilon$ such that it passes through $\xi = 1$ somewhere near midlevel. Fig. 7 summarizes the main effects of increasing $\epsilon$ on the linear stability problem. Fig. 7(a) shows the critical Rayleigh number $Ra_c$ at $k = 1$ for steady and oscillatory convection as a function of $\epsilon$ when $\xi_0 = 0.2$. The figure shows that as $\epsilon$ increases the neutral curve for oscillatory convection terminates on the steady state neutral curve at $\epsilon \approx 2.64$. At the termination point the oscillation frequency vanishes, i.e., this point is a Takens–Bogdanov point. Similar behavior occurs in the formulation used by Weiss et al. [8] as well. Fig. 7(b) shows the location $z_{\text{max}}$ of the maximum of the resulting eigenfunction $|\psi_1|$ as a function of $\epsilon$ for both steady and oscillatory onsets. Note that for steady onset $z_{\text{max}}$ decreases monotonically with $\epsilon$, but appears to saturate for large $\epsilon$, much as $Ra_c$. In contrast for oscillatory onset, $z_{\text{max}}$ first increases before falling towards the steady onset value at $\epsilon \approx 2.64$. These results can be understood by noting that an increase in $\epsilon$ implies an increase in $\xi$ near the lower boundary $z = 0$, or equivalently, a local decrease in the thermal diffusivity. As a result convection becomes easier near the bottom and consequently we expect its maximum amplitude to fall below $z = 1/2$ as $\epsilon$ increases. However, for overstable convection the effect of increasing $\epsilon$ is quite different. This is because increasing $\epsilon$ tends to shift the region susceptible to overstability towards higher locations in the layer. Both of these trends are clearly visible in Fig. 7(b) at small $\epsilon$ although the latter is eventually reversed due to the overall increase in $\xi$ at midlevel which acts to suppress oscillations altogether.
These differences between steady and oscillatory convection manifest themselves in the nonlinear regime as well. Fig. 8(a) shows the Nusselt number $K(\text{Ra})$ for several values of $\epsilon$ when $\zeta_0 = 1$ with $k = 1$. Fig. 8(b) shows the corresponding results for the more interesting case $\zeta_0 = 0.2$. In this case the convective instability is oscillatory and consequently we also show the oscillation frequency $\omega$ (Fig. 8(c)). These figures show that the differences between the effects of variable $\zeta$ on steady and oscillatory convection extend into the nonlinear regime. In the steady case, (Fig. 8(a)) the Nusselt number is quite insensitive to $\epsilon$ and one has to go to Ra values in excess of $10^5$ to see a real difference. Nonetheless, the trend is clear: at each Rayleigh number the Nusselt number decreases monotonically with increasing $\epsilon$. For oscillatory convection the $\epsilon$ dependence is much stronger. As expected the Nusselt number is largest for small $\epsilon$. These solutions also have the smallest asymptotic frequency at large Ra; not surprisingly, this makes convective transport more efficient and hence increases the Nusselt number. In Fig. 9 we show the dependence of the mean temperature profile on $\epsilon$ at a high Rayleigh number in each of these two cases. Observe that as $\epsilon$ increases away from zero the mean temperature profile acquires asymmetry with respect to the midplane $z = 1/2$. For large Rayleigh numbers the layer still develops an isothermal core but now its temperature $T_{\text{core}}$ differs from $T = 1/2$. For steady convection with $\epsilon > 0$, $T_{\text{core}} > 1/2$ while the opposite is the case when $\epsilon < 0$. Consequently the temperature jump in the upper boundary layer is larger than that in the lower one when $\epsilon > 0$ and conversely. Once again the local decrease in the thermal diffusivity near the lower boundary makes convection easier near the bottom and hence we expect its maximum amplitude to fall below $z = 1/2$ as $\epsilon$ increases. In addition, lower thermal diffusivity implies that at a given Rayleigh number the temperature jump across the lower thermal boundary layer also falls below 1/2, as seen in Fig. 9(a). This is ultimately why $T_{\text{core}}$ increases with $\epsilon$. Despite this appealing picture for steady convection, the results for oscillatory convection reveal an opposite trend. Thus, as shown in Fig. 9(b), the core temperature moves towards lower values with increasing $\epsilon$, indicating that
Fig. 8. The Nusselt number $K = Ra/\theta_D$ for several values of $\theta_D$ when $\xi_0 = 0.1$ and $\epsilon = 0.0$ (solid), $\epsilon = 0.5$ (dashed), $\epsilon = 1.0$ (dot-dashed) and $\epsilon = 5.0$ (dot-dot-dot-dashed). (b) Oscillatory convection when $\xi_0 = 0.2$ and $\epsilon = 0.0$ (asterisks), $\epsilon = 2.0$ (diamonds) and $\epsilon = 5.0$ (triangles). (c) Shows the oscillation frequency $\omega$ corresponding to (b). Notice that for steady convection $K(Ra)$ remains largely independent of $\epsilon$, but for oscillatory convection both $K$ and the asymptotic frequency $\omega_{osc}$ depend strongly on $\epsilon$.

despite the decrease in thermal diffusivity near the bottom boundary, the time-averaged temperature drop across the lower boundary layer increases. This is presumably because the repeated flow reversals do, on average, increase the boundary layer thickness, but what is unexpected is the magnitude of the resulting shift in the core temperature. Fig. 10 summarizes the shifts with $\epsilon$ in the core temperatures in the two cases. Fig. 11 shows the profiles of the square of the eigenfunction $|\psi_1(z)|$ in the oscillatory regime when $\epsilon = 0.0$ and $\epsilon = 1.0$, both at onset and at $Ra = 20\times10^6$. When $\epsilon = 0.0$ the profiles are symmetric with largest amplitude at $z = 1/2$. When $\epsilon > 0$ both profiles become asymmetric and peak above midheight; the effect is enhanced by nonlinearity.

The linear theory suggests that oscillations should be present only for $\epsilon \lesssim 2.64$ with steady convection preferred for larger $\epsilon$. Indeed, for these $\epsilon$ values convection sets in as a supercritical steady state bifurcation. However, despite this fact nonlinear oscillations can be found even for $\epsilon > 2.64$ by continuation in $\epsilon$ at finite $Ra$, as shown in Figs. 10 and 12. Fig. 12(c) shows an example computed at $\epsilon = 5.0$ and $Ra = 20\times10^6$ at six equally spaced time intervals during one oscillation period in terms of the magnetic field perturbation. As expected from Eq. (44) the oscillation has zero mean, i.e., at every $(x,z)$ the velocity and magnetic field perturbations oscillate about zero.
Fig. 9. Mean temperature profiles $\bar{T}(z)$ for (a) steady convection and $\epsilon = 0(1)5$ at $Ra = 52100$, $\zeta_0 = 1.0$ and (b) oscillatory convection as a function of $\epsilon = 0(0.5)5$ at $Ra = 20106$, $\zeta_0 = 0.1$, $\sigma = 1.1$. The core temperature shifts monotonically from $T_{core} = 1/2$ as $\epsilon$ increases.

Fig. 10. The core temperature $T_{core}$ determined from high Rayleigh number computations, as a function of $\epsilon$ for steady convection (solid line) when $\zeta_0 = 1.0$ and oscillatory convection (dashed line) when $\zeta_0 = 0.1$, $\sigma = 1.1$. The isothermal core temperature increases slightly with increasing $\epsilon$ in the steady case but decreases in the oscillatory case.

in time. However, because of the large value of $\epsilon$ such oscillations have a pronounced asymmetry with respect to $z = 1/2$, in contrast to Fig. 5(b). Fig. 12(a) shows that this oscillatory branch terminates with decreasing $Ra$ on the steady branch when $Ra \sim 24.1$. The resulting bifurcation diagram (Fig. 12(a)) is therefore similar to that found by Weiss et al. [8] in their study of two-dimensional compressible magnetoconvection. However, there is an important difference as well. Fig. 12(b) shows that the frequency drops to zero as the square root of the distance from the termination point. Thus the bifurcation is not the expected secondary Hopf bifurcation but a finite
Fig. 11. The square of the eigenfunction $|\psi_1(z)|$ for oscillatory convection at onset for (a) $\epsilon = 0.0$ and (b) $\epsilon = 1.0$, showing the gradual development of localized oscillations in the upper part of the layer. (c,d) The corresponding $|\psi_1(z)|^2$ at $Ra = 20\times10^6$. The remaining parameters are $\zeta_0 = 0.1$, $\sigma = 1.1$.

amplitude Takens–Bogdanov point, i.e., in our system the transfer of stability from the steady to the oscillatory branch occurs via a steady state bifurcation of the type usually associated with a parity-breaking bifurcation from a circle of steady states to traveling rolls [24] while the (supercritical) Hopf bifurcation found in Weiss et al. is usually (but not necessarily) associated with the termination of a branch of standing rolls. Of course, our solutions can be used to construct either type of solution (and indeed others as well). However, since our standing rolls always have zero mean they cannot undergo the type of gluing bifurcation that precedes the termination of the SR branch in
Fig. 12. (a) The time-averaged Nusselt number $K$ and (b) oscillation frequency $\omega$ as functions of $Ra$ for $\epsilon = 5.0, \zeta_0 = 0.1, \sigma = 1.1$. (c) The corresponding magnetic field perturbation at $Ra = 20106$, shown as in Fig. 5(b).

a secondary bifurcation [25]. Thus for standing rolls the Takens–Bogdanov point represents the conflation of the gluing bifurcation with the secondary Hopf bifurcation. In either case we expect the oscillatory branch to inherit stability from the branch of steady convection and hence be stable (see Fig. 12a).

Similar results are found when $\zeta$ is increased at fixed $Ra$ for $k = 1$ and various values of $\epsilon$. When $\epsilon = 0$, small amplitude oscillations about the zero state set in provided $\xi < \xi_{TB} \equiv \sigma \pi^2 (1 + \sigma + \sigma \pi^2)^{-1} \approx 0.838$. If we fix the Rayleigh number at a high value ($Ra = 20098$) and increase $\zeta$ we find (Fig. 13 (a)) that the oscillations terminate on the branch of nontrivial steady solutions at $\xi = \xi_c \approx 0.852$. Thus oscillations again persist beyond the linear stability prediction and their termination is again associated with a Takens–Bogdanov point. In particular, the oscillation frequency falls to zero according to $\omega^2 \propto \xi_c - \xi$ (Fig. 13(b,c)).

Bifurcation diagrams similar to Fig. 12(a) arise in systems of ordinary differential equations resulting from a reduction of the partial differential equations of Boussinesq magnetoconvection near the (primary) Takens–Bogdanov point for narrow cells [26]. This problem possesses a reflection symmetry with respect to $\xi = 1/2$, as does ours when $\epsilon = 0$. However, in Rucklidge’s study, the narrowness of the cells is imposed artificially and is
not linked to a large value of the parameter $Q$. As a result, the procedure provides only a local description of the pde dynamics near a particular point in parameter space, in contrast to the global description from our approach. Despite this, Rucklidge finds that the analysis provides a good description of the pde behavior, albeit in a limited parameter regime. Unfortunately as discussed above our description, while global, is also highly degenerate.

6. Highly supercritical convection

It is of considerable theoretical interest to look at the behavior of the (time-averaged) Nusselt number at highly supercritical Rayleigh numbers. For the case of steady convection such an analysis was carried out already by Bassom and Zhang [27]. The overstable case is very similar because the oscillation frequency saturates with increasing Rayleigh number (see Fig. 2). In the following we focus on the case of constant $\zeta$. The complex eigenvalue problem
\[ \frac{D^2 \rho - \rho(D\theta)^2}{\zeta} - \frac{1}{\zeta} \left( \frac{\omega^2 - \omega_0^2}{\sigma} \right) \rho + \frac{\rho K}{\zeta} \left( \frac{\omega^2 + \zeta k^4}{\omega + k^4 + (1/2)\rho} \right) \rho = 0, \]  
\hspace{2cm} (56)

\[ \frac{D(\rho^2 D\theta)}{\zeta} - \frac{\omega k^2}{\zeta} \left( 1 + \frac{\zeta}{\sigma} \right) \rho^2 + \frac{\rho K^2 \omega(1 - \zeta)}{\zeta(\omega^2 + k^4 + (1/2)\rho)} \rho^2 = 0, \]  
\hspace{2cm} (57)

where \( \psi_1 \equiv \rho e^{j\theta} \). In the large Ra limit we expect \( \rho \) and \( K \) to be large, but \( \omega = O(1) \), see Fig. 2. We therefore define

\[ \rho = \rho_0 f(z), \quad Ra K = \lambda \rho_0^2, \quad \omega = \omega_\infty, \]  
\hspace{2cm} (58)

where \( \rho_0 \) measures the maximum value of \( |\psi_1(z)| \), i.e., \( |f(z)| \leq 1 \), and is assumed to be large. Since \( k \) remains fixed the eigenvalue problem becomes

\[ \frac{D^2 f - f(D\theta)^2}{\zeta} - \frac{1}{\zeta} \left( \frac{\omega^2 - \omega_\infty^2}{\sigma} \right) f + \frac{2\lambda (\omega_\infty^2 + \zeta k^4)}{k^6 f} = 0, \]  
\hspace{2cm} (59)

\[ \frac{D(f^2 D\theta)}{\zeta} - \frac{\omega_\infty k^2}{\zeta} \left( 1 + \frac{\zeta}{\sigma} \right) f^2 + \frac{2\lambda \omega_\infty(1 - \zeta)}{\zeta k^4} = 0 \]  
\hspace{2cm} (60)

at leading order in \( \rho_0^{-1} \). These two equations, with suitable boundary conditions, constitute an eigenvalue problem for \( \lambda \) and \( \omega_\infty \). These conditions are (51) by extrapolation to high values of Ra. Both \( \lambda \) and \( \omega_\infty \) rapidly tend to a constant value as Ra is increased and these are the asymptotic values that are shown in Fig. 14(a) as a function of \( \zeta \) for \( k = 1 \). Notice that at \( \zeta \sim 0.858 \) the asymptotic frequency vanishes as the oscillatory branch terminates in a Takens–Bogdanov point on the steady branch. Moreover, for \( \zeta > 0.858 \), \( \lambda \) is constant, since for steady convection the solutions are independent of \( \zeta \).

With the above scaling the contribution from the bulk to \( K^{-1} \) is \( O(\rho_0^{-2}) \) and hence smaller than that from the thermal boundary layers. In the boundary layers the term \( f(D\theta)^2 \) in Eq. (59) is subdominant, as will be verified a posteriori, so that Eq. (59) can be integrated once

\[ \frac{1}{2} (Df)^2 = \frac{1}{2\zeta} \left( \frac{\omega_\infty^2}{\sigma} \right) (1 - f^2) - \frac{2}{k^6} \left( \frac{\omega_\infty^2 + \zeta k^4}{\omega_\infty^2 + k^4} \right) \ln f. \]  
\hspace{2cm} (61)

Thus at leading order,

\[ f = \hat{f} z(-\ln z)^{1/2}, \quad \hat{f} = \frac{2\sqrt{\lambda}}{k^3} \left( \frac{\omega_\infty^2}{\zeta} + k^4 \right)^{1/2}. \]  
\hspace{2cm} (62)

In the following we therefore introduce the boundary layer coordinate

\[ Z = \rho_0 (\ln \rho_0)^{1/2}, \quad Z = O(1). \]  
\hspace{2cm} (63)

It follows that to leading order

\[ K = K_0 \rho_0 (\ln \rho_0)^{1/2}, \quad K_0 = \frac{1}{\pi} \sqrt{\frac{2\lambda}{\zeta}} \left( \frac{\omega_\infty^2 + \zeta k^4}{\omega_\infty^2 + k^4} \right)^{1/2}, \]  
\hspace{2cm} (64)

so that

\[ Ra = \frac{\lambda}{K_0} \rho_0 (\ln \rho_0)^{-1/2}. \]  
\hspace{2cm} (65)
These relations imply a transcendental relation between the Nusselt and Rayleigh numbers when the latter is large:

$$ K = K_1 \text{Ra} \ln(K \text{Ra}), \quad K_1 = \frac{1}{\pi^2 \zeta} \left( \frac{\omega_{\infty}^2 + \zeta k^4}{\omega_{\infty}^2 + k^4} \right), $$

(66)
to leading order. The constant $K_1$ is shown as a function of $\zeta$ in Fig. 14(b), while the relation (66) is compared with numerical solutions to the nonlinear eigenvalue problem (51) in Fig. 15. The figure shows that the asymptotic behavior given by Eq. (66) provides a reasonable fit to the numerical results except that the calculated prefactor $K_1$ is a little too large. This discrepancy indicates that the numerical solutions have not reached values of Ra large
enough for the asymptotics to be completely valid. This is most likely due to the presence of logarithmic terms in the asymptotic expansion. For example, the neglected term in Eq. (59) $f(D0)^2 \approx z^{-1}(-\ln z)^{-3/2}$ to leading order, and this term is indeed small when compared to the $f^{-1}$ term retained in obtaining Eq. (62), but only by a logarithmic factor.

The above analysis applies straightforwardly to the real eigenvalue problem describing steady convection. It suffices to drop the $\theta$ variable and set $\omega_\infty = 0$. Consequently a relation of the form Eq. (66) governs the steady problem as well, as shown by Bassom and Zhang [27], with the constant $K_1$ replaced by $K_1 = 1/\pi^2$. This relation is independent of both $k$ and of the eigenvalue $\lambda$. The prediction that the asymptotic Nusselt number should be independent of the wavenumber is readily confirmed by solving the full eigenvalue problem (51) at large Rayleigh numbers for different values of $k$.

7. Discussion and conclusions

In this paper we have derived by an asymptotic expansion in inverse Chandrasekhar number a reduced set of dynamical equations describing fully nonlinear three-dimensional convection in a strong magnetic field. These equations, summarized in Eqs. (29)–(34), apply to $O(Q)$ Rayleigh numbers and promise to have significant applications in astrophysics and in particular convection in Sunspots, and especially the Sunspot umbra. At these Rayleigh numbers the adopted scaling shows that the thermal forcing is insufficiently strong to generate substantial departures of the magnetic field from the imposed vertical field. It is this property that leads to the simplified formulation described here and hence to the reduced equations that are valid well beyond the regime in which weakly nonlinear theory applies. The resulting equations describe correctly the nonlinear deformation of the mean temperature profile with increasing supercriticality and are to be viewed as describing the dynamics in the bulk, i.e., outside of thin Hartmann boundary layers at the top and bottom. As a result the resolution requirements are substantially reduced and realistic profiles of density and the various diffusivities with height are readily incorporated. We have considered here only the case where $\zeta$ varies linearly with height, decreasing upwards. This choice of variation was motivated by the numerical simulations of two-dimensional compressible magnetoconvection in $m = 1$ polytropic atmospheres by Weiss and colleagues, although our formulation is entirely incompressible. We have found that at large Rayleigh numbers this system develops an isothermal core just as in the case of constant $\zeta$ except that the core temperature shifts away from $T = 1/2$ and that the maximum amplitude of convection is displaced from midlevel. We have seen that for oscillatory convection the dependence on the nonuniformity in $\zeta$, as parameterized by the parameter $\epsilon$, is nontrivial due to two competing effects. First, the assumed $\zeta$ variation tends to localize the oscillations towards the upper boundary. At the same time, however, it tends to raise the value of $\zeta$ at midlevel and hence to suppress oscillations altogether. There is no doubt, however, that the present formulation is capable of describing fully nonlinear oscillations with realistic profiles of $\zeta$, and as such has immediate applicability to the Sunspot problem. Indeed, we have seen that solutions could be obtained for very high Rayleigh numbers, high enough to be essentially in the asymptotic regime. We shall discuss the Sunspot problem in a future publication.

In this paper we have focused instead on understanding the pattern-forming properties of the reduced equations. Remarkably, a number of patterns can be obtained at arbitrary supercriticality by reformulating the reduced equations as a nonlinear eigenvalue problem. In the case of steady convection this eigenvalue problem is real and determines the Nusselt number as an eigenvalue for prescribed values of the Rayleigh number. For oscillatory convection the eigenvalue problem is complex and determines both the time-averaged Nusselt number and the oscillation frequency. In each case the resulting patterns are characterized, at leading order, by a single horizontal wavenumber, and solve the same eigenvalue problem. Thus in the strong field limit the planform approach of Baker and Spiegel [28], Gough et al. [29] and Toomre et al. [30] to problems of this type becomes exact. Of course, should Ra substantially exceed
O(\Omega) values, the dynamical effects of the distorted field would become significant. In such a case the dominance of a single horizontal wavenumber is doubtful; indeed, strong motions can effectively exclude the field from the interior of a cell allowing convection with a smaller horizontal wavenumber than predicted by linear theory. Within the scaling adopted here this is not possible. The solutions of the nonlinear eigenvalue problem with \omega = 0 can be used to construct the familiar rolls, as well as squares, hexagons, regular triangles and a particular rectangular pattern called patchwork quilt, guaranteed to be present at small amplitude. Although all of these patterns have the same Nusselt number in the strong field regime, i.e., they are equally efficient at transporting heat, we anticipate on the basis of the weakly nonlinear theory a preference for roll-like states. However, because of the near-degeneracy among the competing patterns any such selection must occur on a slow time scale. A similar degeneracy characterizes all oscillatory patterns in this regime. Thus in either case the system at large Rayleigh numbers is characterized by a competition between a number of computable nearly degenerate states with weak selection among them. Moreover, the strong influence of the Lorentz force implies that any instabilities lead to states that do not deviate substantially from the computable states. Consequently we expect the statistical properties of the resulting `turbulent’ state to be largely independent of which of the competing states is actually present, although distinction between steady, and traveling or standing patterns is necessary [15]. The slow evolution envisaged above may be viewed as a drift along a manifold of nearly degenerate states. This drift arises because at finite \Omega the zero eigenvalues characterizing the stability of each pattern (and not forced to be zero by symmetry) are in fact nonzero, although much smaller than the eigenvalue describing adjustment in amplitude. These effects are absent from our reduced equations, and we have not considered the resulting slow evolution.

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Appendix A. The nonlinear terms in the streamfunction representation

In the streamfunction representation the nonlinear terms appearing in Eqs. (9)–(12)

\begin{align}
N_\phi (\phi, \psi) &\equiv (\omega \cdot \nabla) w - (u \cdot \nabla) \omega_3, \quad (A.1) \\
M_\phi (A, B) &\equiv (j \cdot \nabla) b_3 - (b \cdot \nabla) j_3, \quad (A.2) \\
N_\psi (\phi, \psi) &\equiv \hat{z} \cdot \nabla \times \nabla \times (\omega \times u), \quad (A.3) \\
M_\psi (A, B) &\equiv \hat{z} \cdot \nabla \times \nabla \times (j \times b), \quad (A.4) \\
M_A (\phi, \psi, A, B) &\equiv \hat{z} \cdot \nabla \times \nabla \times (u \times b), \quad (A.5) \\
M_B (\phi, \psi, A, B) &\equiv (b \cdot \nabla) w - (u \cdot \nabla) b_3, \quad (A.6)
\end{align}

take the form

\begin{align}
N_\phi &= - J[\phi, \nabla_\perp^2 \phi] - J[\nabla^2 \psi, \nabla_\perp^2 \psi] + \nabla_\perp (\nabla_\perp^2 \phi) \cdot \nabla_\perp (\partial_z \psi) - \nabla_\perp (\partial_z \phi) \cdot \nabla_\perp (\nabla^2 \psi) \\
&\quad - \nabla_\perp^2 \psi \nabla_\perp (\partial_z \phi) + \nabla_\perp^2 \phi \nabla_\perp (\partial_z \psi), \quad (A.7)
\end{align}
\[ M_\psi = -J[A, \nabla_\perp^2 A] - J[\nabla_\perp^2 B, \nabla_\perp^2 B] + \nabla_\perp(\nabla_\perp^2 A) \cdot \nabla_\perp(\partial_z B) - \nabla_\perp(\partial_z A) \cdot \nabla_\perp(\nabla_\perp^2 B) \]

\[ -\nabla_\perp B \nabla_\perp^2(\partial_z A) + \nabla_\perp^2 A \nabla_\perp^2(\partial_z B), \quad (A.8) \]

\[ N_\psi = -\nabla^2[J[\phi, \nabla_\perp^2 \psi] + J[\partial_z \phi, \nabla_\perp(\partial_z \psi)] - \nabla_\perp \phi \cdot \nabla_\perp(\partial_z \phi) - \nabla_\perp(\nabla_\perp^2 \psi)] - \partial_z[J[\partial_z \psi, \nabla_\perp^2 \psi] - J[\phi, \nabla_\perp^2 \psi] + \nabla_\perp \phi \cdot \nabla_\perp(\nabla_\perp^2 \psi) + \nabla_\perp(\partial_z \psi) \cdot \nabla_\perp(\nabla_\perp^2 \psi) + \nabla_\perp^2 \psi \nabla_\perp^2(\partial_z \psi) + |\nabla_\perp(\partial_z \phi)|^2 + |\nabla_\perp(\nabla_\perp^2 \psi)|^2 + (\nabla_\perp^2 \phi)^2, \quad (A.9) \]

\[ M_\psi = -\nabla^2[J[A, \nabla_\perp^2 B] + J[\partial_z A, \nabla_\perp^2 B] - \nabla_\perp A \cdot \nabla_\perp(\partial_z A) - \nabla_\perp(\partial_z B) \cdot \nabla_\perp(\nabla_\perp^2 B) - \partial_z[J[\partial_z B, \nabla_\perp^2 A] - J[A, \nabla_\perp^2 A] - 2J[A, \nabla_\perp^2 \psi] + \nabla_\perp A \cdot \nabla_\perp(\nabla_\perp^2 A) + \nabla_\perp(\partial_z A) \cdot \nabla_\perp(\nabla_\perp^2 B) + \nabla_\perp^2 B \nabla_\perp^2(\partial_z A) + |\nabla_\perp(\partial_z A)|^2 + |\nabla_\perp(\nabla_\perp^2 B)|^2 + (\nabla_\perp^2 A)^2, \quad (A.10) \]

\[ M_A = s \nabla^2[J[A, \phi] + J[\partial_z A, \partial_z \phi] + \nabla_\perp A \cdot \nabla_\perp(\partial_z \phi) - \nabla_\perp \phi \cdot \nabla_\perp(\partial_z A)] + \partial_z[J[\phi, \partial_z A] - J[A, \phi] + J[\partial_z \phi, \nabla_\perp^2 \psi] - J[\partial_z B, \nabla_\perp^2 A] + \nabla_\perp^2 B \nabla_\perp^2(\partial_z \phi) + \nabla_\perp \phi \cdot \nabla_\perp(\nabla_\perp^2 B) - \nabla_\perp A \cdot \nabla_\perp(\nabla_\perp^2 \psi) - \nabla_\perp(\partial_z \phi) \cdot \nabla_\perp(\partial_z A) + \nabla_\perp(\partial_z A) \cdot \nabla_\perp(\partial_z \phi), \quad (A.11) \]

\[ M_B = -J[\phi, \nabla_\perp^2 B] + J[A, \nabla_\perp^2 A] + \nabla_\perp(\partial_z \phi) \cdot \nabla_\perp(\nabla_\perp^2 B) - \nabla_\perp(\partial_z A) \cdot \nabla_\perp(\nabla_\perp^2 \psi) - \nabla_\perp^2 \psi \nabla_\perp^2(\partial_z B) + \nabla_\perp^2 B \nabla_\perp^2(\partial_z \psi), \quad (A.12) \]

with the horizontal Jacobian operator \( J[\cdot, \cdot] = \partial_x \cdot \partial_y - \partial_y \cdot \partial_x \).

References