Reduced models for fluid flows with strong constraints

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The presence of a dominant balance in the equations for fluid flow can be exploited to derive an asymptotically exact but simpler set of governing equations. These permit semianalytical and/or numerical explorations of parameter regimes that would otherwise be inaccessible to direct numerical simulation. The derivation of the resulting reduced models is illustrated here for (i) rapidly rotating convection in a plane layer, (ii) convection in a strong magnetic field, and (iii) the magnetorotational instability in accretion disks and the results used to extend our understanding of these systems in the strongly nonlinear regime. © 2007 American Institute of Physics. [DOI: 10.1063/1.2741042]

I. INTRODUCTION

Geophysical and astrophysical flows typically involve a large disparity in characteristic time scales. Fast time scales indicate rapid decay and/or high frequency oscillations and are reflected in the dominance of one or more terms in the equations of motion. For example, many such flows are rotation dominated, a property that is reflected in the balance of the Coriolis term by the pressure term. Other situations may be characterized by strong density stratification, a fact reflected in the large value of the buoyancy frequency. In yet other applications a strong magnetic field may be present, a fact reflected in the large value of the Alfvén speed. Of course, in all these examples the proper measure of the dominant effect requires one to specify the spatial and temporal scales that are of interest. Once these are chosen dimensionless numbers quantifying the above statements may be defined, and these are responsible for the introduction of small (or large) parameters into the equations of motion. In the last several years considerable progress has been made in exploiting the presence of such parameters. In particular, asymptotic techniques have been developed that permit considerable simplification of the governing equations. The equations that emerge, referred to as reduced models, remain fully nonlinear but are in many cases sufficiently simple to permit a semianalytic description of strongly nonlinear flows. In this article we describe some of these developments, focusing on three important problems in geophysics and astrophysics: (i) rapidly rotating convection, (ii) convection in a strong magnetic field, and (iii) the magnetorotational instability (MRI) that is of interest in both laboratory experiments and the theory of accretion disks. All these cases are characterized by the small scale and large anisotropy (appropriately defined) of the resulting flow. In fact, in all the examples we discuss the strong restraint imposed by the dominant force on the flow forces it to be almost two dimensional. Despite this the departures from two dimensionality retain fundamental importance, and the flows remain strongly nonlinear.

The basic idea behind the construction of reduced models is not new. Related techniques have been used extensively in geophysics and have resulted in the highly successful notion of
quasigeostrophy. Related ideas have been used in the theory of Görtler vortices and Rayleigh-Bénard convection. It turns out, however, that this approach can be extended both to regimes and to flows that have hitherto been inaccessible.

II. RAPIDLY ROTATING CONVECTION

Perhaps the simplest example of a flow in the presence of a strong restraint is provided by convection in a plane horizontal layer in rapid rotation about the vertical. This is a problem of fundamental importance in hydrodynamic stability theory and graces the cover of Chandrasekhar’s book on the subject. Chandrasekhar is primarily concerned with linear theory, and since the publication of his book much work has been done on nonlinear aspects of this problem. Unfortunately analytical progress has largely been confined to weakly nonlinear theory, with more vigorous flows accessible only numerically. However, direct numerical simulations in large domains or in the extreme parameter regimes that are typical of geophysical or astrophysical applications. As a result analytical techniques remain valuable, both for purposes of code validation and to reach parameter regimes that remain inaccessible to DNS.

In the following we adopt a Cartesian coordinate system \( x=(x,y,z) \) rotating with constant angular velocity \( \Omega \) about the \( z \) axis, with gravity acting in the negative \( z \) direction, and assume that the fluid density depends linearly on temperature. The resulting dimensionless Boussinesq equations governing the system can be written in the form

\[
D_t \mathbf{u} + \frac{1}{\text{Ro}} \hat{z} \times \mathbf{u} = -\hat{P} \nabla p + \Gamma \hat{z} \mathbf{u} + \frac{1}{\text{Re}} \nabla^2 \mathbf{u},
\]

where \( \mathbf{u}=(u,v,w) \) is the velocity, \( D_t = \partial_t + \mathbf{u} \cdot \nabla \), \( p \) is the pressure, and \( \theta \) is the temperature. The dimensionless numbers \( \text{Ro}, \hat{P}, \text{Pe}, \text{Re}, \) and \( \Gamma \) are defined in terms of dimensional length, velocity, pressure, and temperature scales \( L, U, P, \) and \( \hat{T} \) characterizing the flow: \( \text{Ro} = U/2\Omega L \) is the Rossby number, \( \hat{P} = P/\rho_0 U^2 \) is the Euler number, \( \text{Pe} = U L/\kappa \) is the Péclet number, \( \text{Re} = U L/\nu \) is the Reynolds number, and \( \Gamma = g a \hat{T} L/\kappa^2 > 0 \) is the buoyancy number. Here \( \Omega = |\Omega|, \) \( g \) is the acceleration due to gravity, \( \alpha \) is the coefficient of thermal expansion, \( \nu \) is the kinematic viscosity, \( \kappa \) is the thermal diffusivity, and \( \rho_0 \) is a reference fluid density. We restrict the flow to an unbounded layer of fluid between two impenetrable horizontal lids. Physically relevant boundary conditions include stress-free and/or no-slip mechanical boundary conditions, together with fixed temperature or fixed flux thermal boundary conditions or combinations thereof. In addition, we assume that the rotational Froude number \( \text{Fr}_\Omega = \Omega^2 L/g \ll 1 \), so that the centrifugal buoyancy force is negligible compared to the gravitational buoyancy force. Neglect of the centrifugal buoyancy force \( \Gamma \text{Fr}_\Omega \hat{z} \times (\hat{z} \times \mathbf{x}) \) in the momentum balance requires \( \Gamma \text{Fr}_\Omega \ll 1 \), a constraint that is assumed in what follows.

It is important that the above nondimensionalization remain generic, without a precise definition of the velocity, length, and time scales. In some examples, these scales may be selected by an instability, while in others the scales may be imposed externally by forcing. In the present case we take \( L \) to be the horizontal scale of the flow; this scale may be small, comparable to, or large relative to the layer height \( H \). It is well known that in the case of thermally insulating boundary conditions (the so-called fixed flux convection), linear theory selects a large horizontal scale. In this case the system is described by a single fully nonlinear equation for the temperature, a dramatic simplification of the original problem. This simplification fails, however, for rapid rotation or for \( O(1) \) Biot numbers characterizing the thermal properties of the horizontal boundaries.
In fact, for rapid rotation \( \text{Ro} \ll 1 \) a different type of simplification becomes possible, because the dominant Coriolis force leads to flows that are organized into vertical plumes or columns whose horizontal scale is small compared to the layer height (see Sec. II D). These tall thin structures are usually referred to as Taylor columns.

A. Asymptotic theory for \( \text{Ro} \ll 1 \) and \( A_z \gg 1 \)

The Taylor columns may be laminar or turbulent depending on the strength of the thermal forcing of the layer. Consequently, we must assume that the flow will change on scale \( L \) in the vertical, in addition to the scale \( A_z L \) imposed by the horizontal boundaries. Here \( A_z = H/L \) is the aspect ratio of the flow, and for rapid rotation we expect that \( A_z \gg 1 \). This setup suggests that we introduce a large scale, \( Z = \frac{1}{A_z} z \); in addition, we must also introduce a slow time \( T = A_T^{-1} t \), where \( A_T \gg 1 \). In the following we therefore write\(^{17,18}\)

\[
\begin{align*}
\partial_z &\rightarrow \partial_z + \frac{1}{A_z} \partial_z, \\
\partial_t &\rightarrow \partial_t + \frac{1}{A_T} \partial_T,
\end{align*}
\]  

leading to the rescaled equations

\[
\left( D_t + \frac{1}{A_T} \partial_T + \frac{w}{A_z} \partial_x \right) u + \frac{1}{\text{Ro}} \hat{z} \times u = -\bar{P} \left( \nabla + \frac{\hat{z}}{A_z} \partial_z \right) \bar{p} + \Gamma \theta \hat{z} + \frac{1}{\text{Re}} \left( \nabla + \frac{\hat{z}}{A_z} \partial_z \right)^2 u,
\]

\[
\left( D_t + \frac{1}{A_T} \partial_T + \frac{w}{A_z} \partial_x \right) \theta = \frac{1}{\text{Pe}} \left( \nabla + \frac{\hat{z}}{A_z} \partial_z \right)^2 \theta, \tag{3}
\]

\[
\nabla \cdot u + \frac{1}{A_z} \partial_z \bar{w} = 0.
\]

These equations form the starting point for the derivation of a \textit{closed} set of reduced equations.

We begin by averaging Eqs. (3) over fast temporal and small spatial scales, obtaining

\[
\begin{align*}
\frac{1}{A_T} \partial_T \bar{u} + \frac{1}{A_z} \partial_z (w \bar{u}) + \frac{1}{\text{Ro}} \hat{z} \times \bar{u} &= \left( -\frac{\bar{P}}{A_z} \partial_z \bar{p} + \Gamma \bar{\theta} \right) \hat{z} + \frac{1}{\text{Re} A_z^2} \partial_z^2 \bar{u}, \tag{4a}
\end{align*}
\]

\[
\frac{1}{A_T} \partial_T \bar{\theta} + \frac{1}{A_z} \partial_z (w \bar{\theta}) = \frac{1}{\text{Pe} A_z^2} \partial_z^2 \bar{\theta}, \tag{4b}
\]

\[
\partial_z \bar{w} = 0, \tag{4c}
\]

where the overbar denotes the operation

\[
\bar{f}(Z,T) := \lim_{V \to \infty} \frac{1}{TV} \int_{\tau V} f(x,Z,t,T)dxdt.
\]  

To obtain equations for fluctuating quantities, we write the dependent variables \( v = (u, p, \theta)^T \) in Eq. (3) as a sum of their mean and fluctuating components, i.e.,

\[
v(x,Z,t,T) = \bar{v}(Z,T) + v'(x,Z,t,T), \tag{6}
\]

and subtract the associated mean equations [Eq. (4)]
Next, we expand the dependent variables \( \mathbf{v} = (\tilde{\mathbf{u}}, \mathbf{u}', \tilde{\mathbf{p}}, \mathbf{p}', \tilde{\theta}, \theta')^T \) in terms of the small parameter \( \text{Ro} = \epsilon \),

\[
\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1 + \epsilon^2 \mathbf{v}_2 + O(\epsilon^3),
\]

and, following Ref. 18, choose the scalings

\[
A_T = \epsilon^{-1}, \quad A_Z = \epsilon^{-2}, \quad \Gamma = \epsilon^{-1} \tilde{\Gamma},
\]

where \( \tilde{\Gamma} \), like \( \text{Re} \) and \( \text{Pe} \), is of order one. Examination of the vertical component of Eq. (4a) now reveals that we must take \( \tilde{\mathbf{p}} = \epsilon^2 \), resulting in a hydrostatic balance at leading order. Thus

\[
\partial_z \tilde{\mathbf{p}}_0 = \tilde{\Gamma} \tilde{\theta}_0.
\]

Moreover, the leading-order horizontal components then imply that \( \tilde{\mathbf{u}}_{0,1} = (\tilde{u}_0, \tilde{v}_0, 0) = 0 \). Since Eq. (4c) implies that \( \tilde{\omega} = 0 \), it follows that \( \tilde{\mathbf{u}}_0 = 0 \), resulting in a substantial simplification.

At leading order the mean buoyancy equation now gives

\[
\partial_t \tilde{\theta}_0 + \partial_z (\tilde{w}_0' \theta'_0) = 0,
\]

implying that \( \tilde{w}_0' \theta'_0 = 0 \), a condition that is in general satisfied only when \( w_0' = 0 \) or \( \theta'_0 = 0 \). The former is unphysical in that it requires the small-scale, rotationally constrained flow to be hydrostatic. Therefore, in the following, we take \( \tilde{\theta}'_0 = 0 \), implying that buoyancy fluctuations about the mean temperature profile \( \tilde{\theta}_0(Z) \) are \( O(\epsilon) \).

At \( O(\epsilon) \) the mean and fluctuating buoyancy equations give

\[
\partial_t \tilde{\theta}_0 + \partial_z (\tilde{w}_0' \tilde{\theta}_0') = \frac{1}{\text{Pe}} \tilde{\theta}_0',
\]

and

\[
D_i^0 \theta'_1 + \frac{w_0'}{\text{Pe}} \partial_z \tilde{\theta}_0 = \frac{1}{\text{Pe}} \nabla^2 \theta'_1,
\]

where \( D_i^0 = \partial_t + \mathbf{u}_0' \cdot \nabla \) and we have utilized the fact that \( \tilde{\theta}'_0 = 0 \). At \( O(\epsilon^{-2}) \) the momentum equation yields

\[
\nabla p'_0 = 0,
\]

implying that \( p'_0 = 0 \). With \( \theta'_0 = p'_0 = 0 \), the momentum equation at \( O(\epsilon^{-1}) \) and \( O(\epsilon^0) \) now yields

\[
\tilde{\mathbf{z}} \times \mathbf{u}'_0 = -\nabla p'_1,
\]
Finally, the continuity equation at $O(1)$ and $O(\varepsilon)$ yields

$$\nabla \cdot \mathbf{u}_0^\prime = 0,$$  \hspace{1cm} (17)

$$\nabla \cdot \mathbf{u}_1^\prime + \partial_2 w_0^\prime = 0.$$  \hspace{1cm} (18)

The system [Eqs. (10)–(13) and (15)–(18)] is closed but still quite involved. However, due to its special structure, it is amenable to further simplification, as described next.

1. The Taylor-Proudman constraint and dynamics of the reduced system

Equation (15) indicates that the leading-order flow is in geostrophic balance. Moreover the curl of this equation implies both that $\hat{z} \cdot \nabla \mathbf{u}_0^\prime = 0$ and $\nabla_\perp \cdot \mathbf{u}_0^\prime = 0$ (i.e., the horizontal flow is nondivergent on small scales). Here $\nabla_\perp = (\partial_x, \partial_y, 0)$. Since Eq. (17) then implies that $\partial_2 w_0^\prime = 0$, it follows that

$$\hat{z} \cdot \nabla \mathbf{u}_i^\prime = 0.$$  \hspace{1cm} (19)

Likewise, applying $\hat{z}$ to Eq. (15) and $\hat{z} \cdot \nabla$ to Eq. (13),

$$\hat{z} \cdot \nabla p_1^\prime = 0, \quad \hat{z} \cdot \nabla \theta_1^\prime = 0.$$  \hspace{1cm} (20)

These three relations express the Taylor-Proudman constraint that forces leading-order motion on small spatial scales to be invariant in the direction of the rotation axis. It follows that $\mathbf{u}_0^\prime, p_1^\prime, \theta_1^\prime$ depend on height through the slow variable $z$ only.\footnote{For some thermal boundary conditions a boundary layer may be present at $Z=0, 1$, but this is not the case for fixed temperature boundary conditions.} This is not so automatically for the higher order terms. However, if we interpret the vertical component of Eq. (16) as an equation for $p_2^\prime$, we see immediately that $p_2^\prime$ will grow secularly with the small-scale variable $z$ unless the remaining terms balance. Thus we require the solvability condition

$$D_0^0 w_0^\prime + \partial_2 p_1^\prime = \tilde{\Gamma} \theta_1^\prime + \frac{1}{\text{Re}} \nabla_\perp^2 w_0^\prime.$$  \hspace{1cm} (21)

The horizontal components likewise yield an equation for $w_1^\prime$, and the corresponding solvability condition is

$$D_0^0 (\hat{z} \cdot \nabla \times \mathbf{u}_0^\prime) = \partial_Z w_0^\prime + \frac{1}{\text{Re}} \nabla_\perp^2 (\hat{z} \cdot \nabla \times \mathbf{u}_1^\prime).$$  \hspace{1cm} (22)

These solvability conditions, together with Eqs. (12), (13), and (17), represent the desired reduced system of equations and guarantee that $p_2^\prime$ and $w_1^\prime$ satisfy the Taylor-Proudman constraint as well.

It is instructive at this stage to highlight the differences between geostrophy in the classical small aspect ratio regime\cite{1,2,3,4} and the present large aspect ratio case. In both cases, geostrophy implies horizontally nondivergent leading-order flow, $\nabla_\perp \cdot \mathbf{u}_0^\prime = 0$, and consequently that $\partial_2 w_0^\prime = 0$. However, in the small aspect ratio regime, the strong stable stratification permits weak vertical motions only, i.e., $w_0^\prime = 0$, while in the present large aspect ratio case the unstable stratification permits substantial vertical motions, i.e., $w_0^\prime \neq 0$. These in turn demand weakly divergent horizontal motions at next order, as described by Eq. (18). These considerations do not arise in classical geostrophy.

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\footnotetext[1]{For some thermal boundary conditions a boundary layer may be present at $Z=0, 1$, but this is not the case for fixed temperature boundary conditions.}
2. Stream function formulation

In order to satisfy the continuity conditions (17) and (18) automatically, it is convenient to use a stream function formulation defined by

$$ \mathbf{u}' = -\nabla \times \phi \mathbf{z} - \nabla \times \nabla \times \phi \mathbf{z}. $$

(23)

Since $\mathbf{u}'_0$ is independent of the small scale $z$, it follows that

$$ \mathbf{u}'_0 = (-\partial_y \psi_0, \partial_x \psi_0, \nabla^2 \phi_0), $$

(24)

and hence that $\psi_0 = P \phi$. Thus the pressure perturbation plays the role of the stream function $\psi_0$, much as in standard quasigeostrophy. Equations (21), (22), (13), (12), and (10) become

$$ \partial_t \nabla^2 \phi + J(\psi, \nabla^2 \phi) + \partial_y \psi = \Gamma \theta' + \text{Re}^{-1} \nabla^4 \phi, $$

(25a)

$$ \partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi) - \partial_z \nabla^2 \phi = \text{Re}^{-1} \nabla^4 \psi, $$

(25b)

$$ \partial_t \theta' + J(\psi, \theta') + \nabla^2 \phi \partial_z \bar{\theta} = \text{Pe}^{-1} \nabla^2 \theta', $$

(25c)

$$ \partial_t \bar{\theta} + \partial_z (\bar{\theta} \nabla^2 \phi) = \text{Pe}^{-1} \nabla^2 \bar{\theta}, $$

(25d)

$$ \partial_z \bar{p} = \Gamma \bar{\theta}, $$

(25e)

where $J(f, g) = \partial_t f \partial_x g - \partial_x g \partial_t f$ and the subscripts on $\psi_0$, $\phi_0$, $\theta'$, $\bar{\theta}$, and $\bar{p}$ denote the asymptotic order have been dropped. The absence of the Rossby number from these equations implies a dramatic reduction in resolution requirements, both spatial and temporal, compared with the primitive equations [Eq. (1)], and allows us to explore the behavior of rotating flows at Rossby numbers that are currently inaccessible to DNS.

Equations (25) possess a number of important properties.

- In the absence of dissipation the equations conserve, on the fast time scale, the energy

$$ E = \frac{1}{2} \int_D \left( |\mathbf{u}'_0|^2 + \frac{\Gamma}{\partial_z \bar{\theta}} \theta'^2 \right) dx dy dz, $$

(26)

i.e., $\partial_t E = 0$, and the potential vorticity

$$ \Pi = \nabla^2 \psi + (\nabla^2 \phi) \partial_x \bar{\psi} - \nabla^2 \phi \partial_z \bar{\psi} + \partial_z \bar{\theta} \frac{\theta'}{\partial_z \bar{\theta}}, $$

(27)

i.e., $\partial_t \Pi + J(\psi, \Pi) = 0$.

- The coupling between $\psi$ and $\phi$ occurs through the slow variable $Z$ via stretching due to the Coriolis force in Eq. (25b) and the pressure gradient force in Eq. (25a). The nonlinearity in the momentum equations (25a) and (25b) arise solely through horizontal advection.

- Vertical diffusion of the fluctuating quantities is absent, a consequence of the geometric assumption $A_Z \gg 1$. This reduction in the order of the system in the vertical relegates boundary layer effects to higher order and renders them passive. Consequently, we need only to employ impenetrable boundary conditions on the upper and lower surfaces together with appropriate temperature boundary conditions. When the boundary conditions are no-slip, the velocity boundary layer is of depth $\lambda = O(Re^{3/2} \text{Re}^{-1/2})$ relative to $H$, and its structure can be deduced from the instantaneous interior solution.
• The equations possess an unexpected reflection symmetry,

\[(x, y) \rightarrow (x, -y), \quad (\psi, \phi, \theta', \tilde{\theta}) \rightarrow (-\psi, -\phi, -\theta', \tilde{\theta}),\]

that is absent in the original equations [Eq. (1)]. This extra symmetry is a consequence of the
absence of pseudoscalar terms in Eq. (25), and its consequences are profound. At leading
order, rotationally constrained flows with \( \text{Ro} \ll 1 \) have the same symmetry properties as
nonrotating flows,\(^{19}\) even though at finite Ro this is no longer so; the horizontal velocity
components \( u'_1 = u'_{0,1} + \text{Ro} u'_{1,1} \) in Eq. (24) do not share this symmetry. The full flow does,
therefore, manifest the handedness expected of a rotating system and present in the
Navier-Stokes equation in primitive variables. However, the presence of this asymptotic
symmetry has important consequences for the types of solutions admitted by these equa-
tions and their stability. In particular, in the limit \( \text{Ro} \ll 1 \) the Küppers-Lortz instability\(^{20}\)
is suppressed,\(^{19,21}\) and the preference for cyclonic flows disappears.

B. Rotating Rayleigh-Bénard convection

Equations (25) represent the generic reduced equations. If we choose \( U = \nu / L \) as the velocity
scale for viscous diffusion in the horizontal and \( L^2 / \nu \) for the corresponding time scale, then

\[\text{Ro} = \epsilon, \quad A_Z = \epsilon^{-1}, \quad \Gamma = \frac{e\text{Ra}}{\text{Pr} A_Z}, \quad \text{Re} = 1, \quad \text{Pe} = \text{Pr}.\]

Thus \( \Gamma = e^2 \text{Ra} / \text{Pr} \). We therefore introduce the scaled Rayleigh number \( \text{Ra} = e^2 \text{Ra} \) such that \( \Gamma = \text{Ro} = e^2 \text{Ra} / \text{Pr} \) is of order one. Moreover, one rotation period \( 2\pi / \Omega \) is equivalent to the nondimensional
time \( 4\pi \epsilon \). It turns out that these relations are consistent with the linear theory predictions in
the limit of rapid rotation.\(^{11}\) This property of rapidly rotating convection implies that the reduced
equations describe the system all the way from the critical Rayleigh number to large but finite
values of \( \text{Ra} \). This is a useful property of the present problem but is not a requirement of the
theory.

The resulting equations take the form

\[\partial_t \nabla_1^2 \phi + J(\psi, \nabla_1^2 \phi) + D \psi = \frac{\text{Ra}}{\text{Pr}} \theta' + \nabla_1^4 \phi,\]

\[\partial_t \nabla_1^2 \psi + J(\psi, \nabla_1^2 \psi) - D \nabla_1^2 \phi = \nabla_1^4 \psi,\]

\[\partial_t \theta' + J(\psi, \theta') + \nabla_1^2 \phi D \tilde{\theta} = \text{Pr}^{-1} \nabla_1^2 \theta',\]

\[\partial_t \tilde{\theta} + D (\theta' \nabla_1^2 \theta) = \text{Pr}^{-1} D^2 \tilde{\theta},\]

\[\partial_2 \tilde{\theta} = \frac{\text{Ra}}{\text{Pr}} \theta',\]

where \( D = \partial_2 \), and are studied here with the boundary conditions

\[\phi |_{Z=0} = \phi |_{Z=1} = 0, \quad \theta' |_{Z=0} = \theta' |_{Z=1} = 0, \quad \tilde{\theta} |_{Z=0} = 1, \quad \tilde{\theta} |_{Z=1} = 0,\]

corresponding to impenetrable, fixed temperature boundaries. The governing equations then imply that
leading to a natural association at leading order with stress-free boundary conditions. As noted earlier, other velocity boundary conditions lead to passive boundary layers [see, for instance, the $O(e^{3/2})$ Ekman boundary layer associated with no-slip boundaries], which can be computed a posteriori once the interior solution is known.

Figure 1(a) shows the threshold (solid line) for the onset of steady convection in the Rayleigh number ($Ra$) versus Taylor number ($Ta = 4e^\phi$) parameter space obtained from the unscaled equations. For $Pr < Pr^* = 0.676 605$, the above bifurcation is preceded by the onset of overstability, a term used in fluid mechanics to refer to a Hopf bifurcation. Also shown are the (dotted) lines of constant convective Rossby number $Ro_{conv} = e^\phi \sqrt{Ra/Pr}$ for $Pr = 7$. This number is a precise a priori measure of the importance of rotation as $Ra$ is varied, and remains small even when $Ra$
over, for statistically steady states the accumulation of averages in
we can determine the associated mean temperature profile as a function of
simple spatial averaging, a procedure supported by the observation that
unstable mode. In view of the extra symmetry
squares, hexagons, regular triangles, and a pattern of rectangles called the patchwork quilt.31,32 For
these solutions
interior develops as \( Ra \rightarrow \infty \). Of course, once the Rayleigh number becomes so large that the flow
is no longer strongly constrained \( Ra^{-1} \approx o(\epsilon^4) \); region 4 of Fig. 1(b)], these scaling laws no longer
apply. Similar results hold in the overstable case as well.19 Figure 2 shows representative solutions
for the single
D. Simulations of the reduced equations
The reduced equations are substantially simpler to integrate than the primitive equations. To
perform the simulations we set \( \partial_x \bar{\theta} = 0 \) and replace the spatiotemporal averaging over \( x,y,t \) by a
simple spatial averaging, a procedure supported by the observation that \( \partial_x \bar{\theta} \rightarrow 0 \) as \( t \rightarrow \infty \). Moreover,
for statistically steady states the accumulation of averages in \( t \) becomes equivalent to hori-
zontally averaging across rising and falling plumes.17 Accordingly, for the simulations we replace
Eq. (30d) by
\[ D\bar{\theta} = -1 + Pr(\bar{\theta}'\nabla_z^2\phi - \langle \bar{\theta}'\nabla_z^2\phi \rangle_Z), \]  

where the overbar now denotes horizontal spatial averaging only, and \( \langle \cdot \rangle_Z \) denotes the operation

\[ \langle f \rangle_Z = \int_0^1 f dZ. \]

As a consequence of this approximation the results below are only valid in the statistically steady state regime.

Figure 3 shows grayscale cross sections of the vertical vorticity field \( \omega'_z \) at the lower boundary \( Z=0 \) for different values of \( Ra \) and \( Pr \). The figure reveals the presence of three distinct regimes we refer to as the Taylor column regime, the plume regime, and quasigeostrophic turbulence. The Taylor columns are found in the range \( Ra \leq 40 \) when \( Pr=7 \) and for \( Ra \leq 80 \) as \( Pr \to \infty \), and are characterized by an extraordinary degree of coherence in the vertical. The structure of these columns is symmetric about the midplane in \( \theta' \) and \( w' \), and antisymmetric in \( \omega'_z \). In addition, these plots reveal the presence of shielding, with vortices consisting of a cyclonic (or anticyclonic) core surrounded by anticyclonic (or cyclonic) sleeve extending throughout the layer depth. Because of
this shielding the vortices interact only weakly, behaving like dilute particles with zero circulation. Figure 4 shows volume renderings of temperature in this columnar regime obtained from DNS and laboratory experiments. At larger Rayleigh numbers the Taylor columns lose stability and become plumelike with finite net circulation; as a result the plumes interact much more strongly. Finally, at even larger Rayleigh numbers the plumes lose their identity and geostrophic turbulence ensues. Figure 5 shows the resulting profiles of the mean temperature \( \bar{T} \) and its midplane gradient \( -D\bar{T}/Dz \) as a function of \( Ra \). The profiles reveal that the turbulent states are less efficient at heat transport than the single mode solutions [cf. Fig. 2(b)] and that this difference increases with \( Ra \). Indeed Fig. 5(c) shows that, depending on the Prandtl number, the mean temperature gradient saturates with increasing Rayleigh number; this behavior can be used to define the onset of the geostrophic turbulence regime.34

The above results differ qualitatively from nonrotating turbulent convection in which the core becomes more and more isothermal as the Rayleigh number increases and are at present inaccessible to DNS of the primitive equations. However, experiments with \( Ro < 0.2 \) or less23,35 do confirm the presence of equal populations of cyclonic and anticyclonic vortices revealed by the asymptotic analysis for \( Ro \approx 1 \).

III. CONVECTION IN A STRONG MAGNETIC FIELD

The study of convection in an imposed magnetic field is largely motivated by the observed magnetic field dynamics in the solar convection zone.36 We describe here a recent development that allows us to extend existing results37,38 semianalytically into the fully nonlinear regime. The resulting solutions are valid at Rayleigh numbers far above onset and are constructed via an...
asymptotic expansion in inverse powers of the Chandrasekhar number $Q = B_0^2 H^2 / \mu_0 \rho \eta \nu$ that measures the strength of the imposed magnetic field $B_0$. Here $\eta$ is the Ohmic diffusivity, $\rho$ the
fluid density, and \( H \) the layer depth. As in the problem of rapidly rotating convection, the dominant nonlinearity at large \( Q \) arises from the nonlinear distortion of the mean temperature profile; the strong magnetic field resists distortion by the velocity field and the Lorentz force arising from the distortion of the magnetic field remains small. As a result our solutions are characterized by a single wave number in the horizontal with the vertical structure given by the solution of a nonlinear eigenvalue problem for the Nusselt number, much as for rapidly rotating convection. We consider both vertical magnetic fields (such as those occurring in the Sunspot umbra) and inclined fields (such as those occurring in the penumbra).

Of particular interest is the fact that our approach applies equally to more realistic situations in which the background state depends on depth. Such dependence can arise in a number of ways. Here we focus on two possibilities, that the magnetic Prandtl number \( \xi \) depends nontrivially on the depth \( z \) within the layer and that the layer is stratified. Both of these effects destroy the midplane reflection symmetry that is otherwise present in the equations, with interesting consequences for high Rayleigh number magnetoconvection. For astrophysical implications see Refs. 39–41.

### A. Governing equations

The dimensionless Boussinesq equations describing magnetoconvection in a plane horizontal layer with an imposed vertical magnetic field \( B_0 \) are

\[
\frac{1}{Pr} D_t \mathbf{u} = -\nabla \pi + \xi Q \mathbf{B} \cdot \nabla \mathbf{B} + Ra \theta \hat{z} + \nabla^2 \mathbf{u},
\]

\[
D_t \theta = \nabla^2 \theta,
\]

\[
D_t \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} - \nabla \times (\xi \nabla \times \mathbf{B}),
\]

with \( \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0 \). Here \( \mathbf{u} = (u, v, w) \) is the velocity field in Cartesian coordinates \((x, y, z)\) with \( z \) vertically upward, \( \mathbf{B} = \hat{z} + \mathbf{b} \), where \( \mathbf{b} \) is the magnetic field perturbation, \( \theta \) denotes the temperature, and \( \pi \) is the dimensionless total (thermal and magnetic) pressure. In contrast to Sec. II the equations have been nondimensionalized with respect to the thermal diffusion time in the vertical. The magnetic Prandtl number \( \xi = \eta / \kappa \) varies with depth due to depth-dependent Ohmic diffusivity \( \eta \).

These equations are solved for a conducting fluid confined between impenetrable boundaries at fixed temperatures

\[
\theta(0) = 1, \quad \theta(1) = 0,
\]

with periodic boundary conditions in the horizontal. The remaining boundary conditions are unimportant in the limit of large \( Q \). In the following we focus on \( O(Q^{-1/4}) \) horizontal scales and \( O(Q^{1/2}) \) oscillation frequencies, i.e., we write

\[
\partial_x, \partial_y \rightarrow Q^{1/4} (\partial_x, \partial_y), \quad \partial_z \rightarrow Q^{1/2} \partial_z + \partial_T,
\]

where \( T = Q^{-1/2} t \), and consider \( O(Q) \) Rayleigh numbers, i.e., we also write \( Ra = Q \overline{Ra} \). In this regime the convective motions do not distort the field substantially and the dominant nonlinear effect arises from the distortion of the mean temperature profile. To proceed we define the stream functions \( \psi, \phi, A, B \) using

\[
\mathbf{u}(x, y, z, t) = -\nabla \times \psi(x, y, Z, t) \hat{z} - \nabla \times \nabla \times \phi(x, y, Z, t) \hat{z},
\]

\[
\mathbf{b}(x, y, z, t) = -\nabla \times A(x, y, Z, t) \hat{z} - \nabla \times \nabla \times B(x, y, Z, t) \hat{z},
\]

and scale them according to
\[ \psi = \psi_1(x,y,Z,t,T) + O(Q^{-1/4}), \quad \phi = Q^{-1/4}\psi_1(x,y,Z,t,T) + O(Q^{-1/2}), \]  
\[ A = Q^{-1/2}A_1(x,y,Z,t,T) + O(Q^{-3/4}), \quad B = Q^{-3/4}B_1(x,y,Z,t,T) + O(Q^{-1}), \]  
\[ \theta = \theta_1(Z,T) + Q^{-1/4}\theta_1(x,y,Z,t,T) + Q^{-1/2}\theta_2(x,y,Z,t,T) + O(Q^{-3/4}). \]

In these expressions \( x, y, t \) are the scaled fast variables; \( Z \equiv z \) and \( T \) are slow variables. With the above scaling the velocity \( u \) and magnetic field perturbation \( b \) are \( O(Q^{1/4}) \) and \( O(Q^{-1/4}) \), respectively. Thus the convective velocities are substantial [the convective kinetic energy \( E \) in a convection cell is \( O(1) \)] but the magnetic field remains primarily vertical.

**B. The reduced equations**

The above equations can be solved by an asymptotic expansion in powers of \( Q^{-1/4} \), as described in Refs. 40 and 42. It follows that

\[ \frac{1}{\Pr} [\partial_t \nabla^2 \psi_1 + J(\psi_1, \nabla \psi_1)] = \zeta(Z)D\psi_1 + \zeta(Z)J(A_1, \nabla \phi_1) + \nabla \phi_1 + O(Q^{-1/4}), \]

\[ \frac{1}{\Pr} [\partial_t \nabla^2 \phi_1 + J(\psi_1, \nabla \phi_1)] = Ra\theta_1 + \zeta(Z)D\nabla \phi_1 + \zeta(Z)J(A_1, \nabla \phi_1) + \nabla \phi_1 + O(Q^{-1/4}), \]

\[ \partial_t A_1 + J(\psi_1, A_1) = D\psi_1 + \zeta(Z)\nabla \phi_1 + O(Q^{-1/4}), \]

\[ \partial_t B_1 + J(\psi_1, B_1) - J(A_1, \nabla \phi_1) = D\nabla \phi_1 + \zeta(Z)\nabla \phi_1 + O(Q^{-1/4}), \]

where \( D = \partial_Z \). Moreover, the temperature equation yields at \( O(Q^{1/4}) \),

\[ \partial_t \theta_1 + J(\psi_1, \theta_1) - \nabla \phi_1 = - \nabla \phi_1 D \bar{\theta}_0, \]

and at \( O(1) \),

\[ \partial_t \theta_2 + \partial_t \bar{\theta}_0 + J(\psi_1, \theta_2) - \nabla \phi_1 D \phi_1 \cdot \nabla \phi_1 + \nabla \phi_1^2 \phi_1 D \theta_0 + \nabla \phi_1^2 \phi_2 D \bar{\theta}_0 = \nabla \phi_1 D \theta_2 + D^2 \bar{\theta}_0. \]

The solvability condition for the mean part of \( \theta_2 \) now yields

\[ \partial_t \bar{\theta}_0 + D(\nabla^2 \phi_1 \bar{\theta}_0) = D^2 \bar{\theta}_0. \]

Equations (52) and (54) together with Eqs. (48)–(51) form a closed system of reduced equations describing magnetoconvection in the presence of a strong vertical magnetic field. Noticeably, the \( (\psi_1, A_1) \) equations decouple. Therefore the resulting equations possess an invariant subspace \( \psi_1 = A_1 = 0 \), and for small amplitudes both \( \psi_1 \) and \( A_1 \) decay to zero. This simplification does not arise when the imposed field is oblique. At larger amplitudes it is possible, however, that nonzero \( \psi_1, A_1 \) are spontaneously generated. It is likely that numerical integration of the reduced system will provide valuable information about the dynamics of fully nonlinear convection in a strong vertical magnetic field, much as already discussed in Sec. II.

Single mode solutions of Eqs. (49), (51), (52), and (54) follow on writing

\[ (\phi_1, B_1, \theta_1) = \frac{1}{2} \Phi(Z, B(Z), \Theta(Z))h(x,y)e^{iut} + \text{c.c.}, \]

where \( h(x,y) \) again satisfies the planform equation [Eq. (33)]. Then
while Eq. (54) implies that

\[ D \hat{\theta}_0 \left[ 1 + \frac{k^6}{2(\omega^2 + k^4)} |\Phi|^2 \right] = - \overline{\text{Nu}}, \]  

where

\[ \overline{\text{Nu}} = \left[ \int_0^1 \frac{\omega^2 + k^4}{\omega^2 + k^4 + (1/2)k^6 |\Phi|^2} dZ \right]^{-1}. \]  

Combining the above expressions, we obtain the nonlinear eigenvalue problem

\[ D^2 \Phi - \frac{(D_\zeta)^2 k^2}{i \omega + \zeta k^2} D \Phi - \frac{1}{\zeta^2 \cos^2 \theta} \left( \frac{i \omega}{\text{Pr}} + k^2 \right)(i \omega + \zeta k^2) \Phi + \frac{\text{Ra} \overline{\text{Nu}}}{\zeta \cos^2 \theta} \frac{(i \omega + \zeta k^2)(-i \omega + k^2)}{\omega^2 + k^4 + (1/2)k^6 |\Phi|^2} k^2 \Phi = 0. \]  

This problem is to be solved subject to the boundary conditions

\[ \Phi(0) = \Phi(1) = 0, \]  

imposing impermeability of the boundaries. These boundary conditions are independent not only of the velocity boundary conditions at the top and bottom but also of the details of the magnetic boundary conditions. Consequently, the solutions of the nonlinear eigenvalue problem describe the solutions in the bulk of the layer, outside thin passive Hartmann boundary layers next to the boundaries.\(^{42}\)

The solution of Eqs. (59) and (60) determines the quantities \(\text{Ra} \overline{\text{Nu}}\) and \(\omega\) as eigenvalues; the associated eigenfunction \(\Phi\) can then be used to calculate \(\overline{\text{Nu}}\) and hence \(\text{Ra}\). Note that steady solutions \((\omega = 0)\) are independent of both \(\text{Pr}\) and \(\zeta\). The latter is not generally true, and for finite \(Q\) the steady solutions do depend on \(\zeta\).

In the case of an oblique magnetic field \(\mathbf{B}_0 = (\sin \vartheta, 0, \cos \vartheta) \mathbf{B}_0\), the derivation requires the inclusion of a fast scale in the vertical, in addition to the scale \(Z\) imposed by the horizontal boundaries. The corresponding derivation can be carried out either in Fourier space\(^{40}\) or using nonorthogonal coordinates in physical space,\(^{18}\) with the result

\[ D^2 \Phi - \frac{(D_\zeta)^2 k^2}{i \omega + \zeta k^2} D \Phi - \frac{1}{\zeta \cos^2 \vartheta} \left( \frac{i \omega}{\text{Pr}} + k^2 \right)(i \omega + \zeta k^2) \Phi + \frac{\text{Ra} \overline{\text{Nu}}}{\zeta \cos^2 \vartheta} \frac{(i \omega + \zeta k^2)(-i \omega + k^2)}{\omega^2 + k^4 + (1/2)k^6 |\Phi|^2} k^2 \Phi = 0, \]  

subject again to Eq. (60). Here

\[ \overline{\text{Nu}} = \left[ \int_0^1 \frac{\omega^2 + k^4}{\omega^2 + k^4 + (1/2)k^6 |\Phi|^2} dZ \right]^{-1}, \]  

and the mean temperature \(\overline{\theta}_0\) is given by

\[ D \overline{\theta}_0 \left[ 1 + \frac{1}{2} \frac{k^2 |\Phi|^2}{\omega^2 + k^4} \right] = - \overline{\text{Nu}}. \]  

The overbar now includes averaging over the fast variable in the \(z\) direction. For the so-called perpendicular rolls, with axes perpendicular to both \(\mathbf{z}\) and \(\mathbf{B}_0\), the horizontal wave number \(k_\perp = k \cos \vartheta\).
C. Results

Solutions to the above eigenvalue problems depend on the prescribed function $\zeta(Z)$ as well as the parameters $Ra$, $k$, $k_\perp$, and $Pr$. We present first the results for constant $\zeta$, followed by the case in which $\zeta=1$ somewhere in the layer, implying a preference for oscillatory convection in one part of the layer ($\zeta<1$) and for steady convection in the rest ($\zeta>1$). We discuss both vertical ($\vartheta=0$) and oblique ($\vartheta \neq 0$) magnetic fields but only for perpendicular rolls. All results are obtained with the horizontal wave number $k_\perp=1$ and $Pr=1.1$.

1. Constant $\zeta$

In Fig. 6 we show the (time-averaged) Nusselt number $\overline{Nu}$ for both steady and oscillatory convection for several values of $\vartheta$ as a function of the scaled Rayleigh number $\overline{Ra}$. Observe that solutions can be obtained for highly supercritical Rayleigh numbers and that $\overline{Nu}$ increases monotonically with increasing $\overline{Ra}$ while the frequency appears to saturate. For moderate values of $\vartheta$ the temperature gradients are confined to thinner and thinner boundary layers at the top and bottom as $\overline{Ra}$ increases; this process is more effective for steady convection but occurs in the oscillatory case as well. At the same time the bulk of the layer becomes more and more isothermal (Fig. 7). Midplane symmetry implies that these boundary layers are identical and that the isothermal interior has temperature $\overline{\theta}_0=1/2$.

With increasing tilt of the field both steady and oscillatory convection become less efficient at transporting heat, and the Rayleigh number dependence of the Nusselt number becomes weaker (Fig. 6). This is due to an increase in the Lorentz force, which in turn reduces the amplitude of convection. The resulting dependence on the tilt angle is much stronger in the oscillatory regime since Ohmic diffusion now has only a finite time to reduce the Lorentz force due to field distortion before the flow reverses. In contrast, in the steady case the Lorentz force exerts a much weaker

![Graph](image_url)

**FIG. 6.** (a) The (time-averaged) Nusselt number $\overline{Nu}$ for steady (dashed lines) and oscillatory (solid lines) convection for $\vartheta=0, 10^\circ, 20^\circ, 30^\circ$ as a function of the scaled Rayleigh number $\overline{Ra}$ when $\zeta=0.1$ and $Pr=1.1$. (b) The corresponding oscillation frequency $\omega$. 

effect and the reduction of the Nusselt number is largely due to a geometrical effect: the strong oblique magnetic field inclines the convection cells relative to the vertical, allowing them more time to lose their upward buoyancy to adjacent descending fluid.

Figure 8 shows the corresponding results for the oscillatory mode when $\theta = \pi/4$. The figure reveals a remarkable behavior: the Nusselt number $\text{Nu}$ initially increases rapidly with $\text{Ra}$ as in the vertical magnetic field case, but then undergoes a hysteretic transition to a new state characterized by a small Nusselt number, and one that decreases slowly with increasing $\text{Ra}$. As this state is followed to larger Rayleigh numbers, we see that the mean temperature becomes almost piecewise linear [Fig. 9(a)], with a limited isothermal core. The extent of this core quickly saturates, in contrast to the case of a vertical field for which the isothermal core grows continuously with $\text{Ra}$ as the temperature gradients are compressed into ever thinner thermal boundary layers [Fig. 7].

Evidently, in this state increasing the heat input does not result in increased heat transport across the layer. Instead, the added energy is all stored in the magnetic field perturbations since the field strength is large, this is achieved with small deformation of the field [Fig. 9(b)], thereby reducing the transport of heat across the layer. In this regime (i.e., on the branch where the Nusselt number remains low as $\text{Ra}$ is increased) the system of perpendicular rolls therefore behaves much more like one with an imposed horizontal field (cf. Ref. 43).

2. Variable $\zeta$

We now discuss the effect of prescribing a depth-dependent diffusivity, $\zeta(Z) = \zeta_0 + \epsilon(1-Z)$, while neglecting the effects of stratification.

Figure 10(a) shows the Nusselt number $\overline{\text{Nu}}(\overline{\text{Ra}})$ for a vertical field and several values of $\epsilon$ when $\zeta_0 = 1$. Figure 10(b) shows the corresponding results for the more interesting case $\zeta_0 = 0.2$. In this case the convective instability is oscillatory and consequently we also show the oscillation frequency $\omega$ [Fig. 10(c)]. In the steady case [Fig. 10(a)], the Nusselt number is quite insensitive to $\epsilon$ and one has to go to $\overline{\text{Ra}}$ values in excess of $10^5$ to see a real difference. Nonetheless the trend is clear: at each Rayleigh number the Nusselt number decreases monotonically with increasing $\epsilon$. For oscillatory convection the $\epsilon$ dependence is much stronger. As expected the Nusselt number is largest for small $\epsilon$. These solutions also have the smallest asymptotic frequency at large $\overline{\text{Ra}}$; not surprisingly this makes the convective transport more efficient and hence increases the Nusselt number.
number. In Fig. 11 we show the development of asymmetry in the mean temperature profile with increasing $\varepsilon$. For steady convection with $\delta=\pi/4$ and $\zeta=0.1$, $\nu=1.1$, one finds that $T_{\text{core}}<1/2$, i.e., the temperature jump in the upper boundary layer is larger than that in the lower one. However, as shown in Fig. 11(b), the opposite is the case when convection is oscillatory. Theoretical predictions of $T_{\text{core}}$ are not available.

Finally, nonlinear oscillations may be present even when linear theory predicts steady onset. Figure 12 shows that in this case the oscillation frequency decreases to zero at finite amplitude, where the branch of oscillatory solutions bifurcates from the steady branch.

IV. VARIATIONS ON A THEME

This section is devoted to modifications and extensions of the above approach and focuses on the inclusion of large-scale modulation in the horizontal, the effect of background stratification and an application of the approach to another type of fingering instability, the MRI.
A. Large-scale modulation in the horizontal

Spatial modulation on scales larger than the scale of the basic cells or fingers may lead to large-scale instabilities of the patterns described above; related equations can be used to incorporate large-scale spatial inhomogeneities into the theory.

In the case of rapidly rotating convection, the inclusion of the large scales $X = A_X^{-1} x$, $Y = A_Y^{-1} y$, in addition to the fast variables $x$ and $y$, results in the large-scale equations

\[
\frac{1}{A_T} \partial_T \tilde{u} + \frac{1}{A_X} \tilde{u} \cdot \nabla_X \tilde{u} + \frac{1}{A_Z} \partial_Z \tilde{u} + \frac{1}{A_X} \nabla_X \cdot (\tilde{u}' \tilde{u}') + \frac{1}{A_Z} \partial_Z (\tilde{w}' \tilde{w}') + \frac{1}{Ro} \hat{z} \times \tilde{u} = - \bar{P} \left( \frac{1}{A_X} \nabla_X + \frac{\hat{z}}{A_Z} \partial_Z \right) \bar{p} + \Gamma \hat{\theta} \tilde{z} + \frac{1}{Re} \left( \frac{1}{A_X} \nabla_X + \frac{\hat{z}}{A_Z} \partial_Z \right)^2 \tilde{u},
\]

FIG. 9. (a) Mean temperature profiles $\bar{\theta}_a(Z)$ and (b) the convection amplitude as measured by $|\Phi(Z)|$ for oscillatory perpendicular rolls at $\theta = \pi/4$ and several values of the (scaled) Rayleigh number showing the development of broad boundary layers of approximate thickness $1/2\mathrm{Nu}$ and a small isothermal core with increasing $Ra$ when $\zeta = 0.1$, $Pr = 1.1$. These properties are characteristic of the "horizontal" convection mode.
\[
\frac{1}{A_T} \partial_T \bar{\theta} + \frac{1}{A_X} \bar{u} \cdot \nabla_X \bar{\theta} + \frac{1}{A_Z} \bar{w} \partial_Z \bar{\theta} + \frac{1}{A_X} \nabla_X \cdot (\bar{u} \, \bar{\theta}') + \frac{1}{A_Z} \partial_Z (\bar{w} \, \bar{\theta}') = \frac{1}{P_v} \left( \frac{1}{A_X} \nabla_X + \frac{1}{A_Z} \partial_Z \right)^2 \bar{\theta},
\]

where \( \nabla_X = (\partial_X, \partial_Y, 0) \) [cf. Eq. (4)]. Reduced equations can be deduced using an asymptotic expansion in powers of \( \epsilon^{1/2} \) [cf. Eq. (8)] in conjunction with the scaling assumptions in Eq. (9) and \( \hat{P} = \epsilon^{-2}, \hat{A}_X = \epsilon^{-3/2} \). From the leading-order momentum and continuity equations, one again deduces that the layer is in hydrostatic balance, described by Eq. (10), together with \( \bar{u}_0 = 0 \). In addition Eq. (64c) implies that \( \bar{w}_{1/2} = 0 \). At \( O(\epsilon^{1/2}) \) the presence of large-scale lateral inhomogeneity leads to the geostrophic balance

\[
\hat{z} \times \bar{u}_{1/2} = -\nabla_X \bar{\rho}_0,
\]

implying that \( \nabla_X \bar{u}_{1/2} = 0 \). The averaged continuity equation then implies that \( \bar{w}_1 = 0 \) as well. As a result the large-scale variables \( \bar{u}_{1/2}, \bar{\rho}_0, \bar{\theta}_0 \) may be written in terms of the geostrophic stream function \( \bar{\Psi}_0(X, Y, Z, T) \), namely,

\[
\bar{\rho}_0 = \bar{\Psi}_0, \quad \bar{u}_{1/2} = -\nabla_X \times \bar{\Psi}_0 \hat{z}, \quad \bar{\theta}_0 = \bar{\Psi}_0 \hat{z} \cdot \partial_Z \bar{\Psi}_0.
\]

The prognostic equation for \( \bar{\Psi}_0 \) is deduced at \( O(\epsilon^2) \) from the mean temperature equation.
Here we have used the results \( \theta_0'/\theta_{1/2}' = 0 \) that follow from Eq. (64b) at \( O(\varepsilon) \) and \( O(\varepsilon^{3/2}) \), respectively [cf. Eq. (11)]. It can also be checked that a similar expansion of the fluctuating equations again leads to Eqs. (25a)–(25c); together with Eqs. (66) and (67) these equations form a closed system.

In the single mode case Eq. (67) takes the form

\[
\bar{\vartheta} \theta_0 + \bar{\mathbf{u}}_{1/2} \cdot \nabla \bar{\theta}_0 + \partial_Z \left( \frac{1}{\text{Pe}} + \text{Pe} k^2 \Phi \right) \partial_Z \bar{\theta}_0 = 0.
\]  

(68)

where

\[
\partial_Z^2 \Phi - \frac{1}{\text{Pe}} \partial_Z \Phi - \frac{1}{\text{Pe}} \partial_Z \Phi \partial_Z \bar{\theta}_0 = 0.
\]  

(69)

A similar system can be derived in the case of overstable convection. Note that the inclusion of the time scale \( T \) as well as of the large spatial scales \( X, Y \) now prevents the integration of Eq. (68), and hence the introduction of the Nusselt number as a constant of integration. The properties of this
coupled system and, in particular, the stability properties of the single mode state within this system have not been studied.

B. Magnetoconvection in a stratified atmosphere

Background stratification is not only important for astrophysical applications but also provides another source of asymmetry with respect to the vertical direction. In such circumstances we expect qualitatively similar results to those found for depth-dependent $\xi$. We eliminate sound waves using the magnetoanelastic approximation \cite{46,47} and thereby focus on dynamics on time scales long compared with the sound travel time. Given the layer depth $d$ and characteristic density $\rho_s$, temperature $T_s$, and pressure $p_s[=R\rho_sT_s]$ at $z=0$, an isentropic basic state is described by the nondimensional equations

$$
\frac{dp_0}{dz} = -\frac{\gamma}{\gamma-1} \delta_\xi \rho_0, \quad p_0 = \rho_0 T_0,
$$

where

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig12.png}
\caption{(a) Time-averaged Nusselt number $\overline{Nu}$ and (b) oscillation frequency $\omega$ as functions of $\overline{Ra}$ for $\delta=0$, $\xi_0=0.2$, $\epsilon=5$, Pr=1.1 (dashed lines). Solid lines show the corresponding results for steady convection.}
\end{figure}
\[ \delta_T = \frac{g d}{c_p T_s} \]

is the inverse (temperature) scale height at \( \varepsilon = 0 \) (\( dT_0/dz = -\delta_T \)), \( \mathcal{R} = c_p - c_v \) is the gas constant, and \( \gamma = c_p/c_v \) denotes the ratio of specific heats at constant pressure and volume. It follows that

\[ T_0 = 1 - \delta_T \varepsilon, \quad p_0 = (1 - \delta_T \varepsilon)^{1/(\gamma - 1)}, \quad \rho_0 = (1 - \delta_T \varepsilon)^{\gamma/(\gamma - 1)}. \]

The dimensionless magnetoanelastic equations then become

\[ D_t \mathbf{u} = -\frac{1}{\rho_0} \nabla p_1 - G \mathcal{R} \frac{\rho_1}{\rho_0} \hat{z} + \frac{Q}{\rho_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{\sigma}{\rho_0} \nabla \cdot \tau, \]

\[ \nabla \cdot \rho_0 \mathbf{u} = 0, \]

\[ \rho_0 T_0 D_t S_1 = \nabla \cdot (K_0 \nabla T_1) + \frac{\xi Q \delta_T}{\mathcal{R}} \nabla \times \mathbf{B}^2 + \frac{\sigma \delta_T}{\mathcal{R}} \tau_{ij} \partial_i \mathbf{u}_j, \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \xi \nabla \times (\mathbf{u}_0 \nabla \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0, \]

where \( \tau = \mu_0 (\partial_i \mathbf{u}_j + \partial_j \mathbf{u}_i - 2/3 \delta_{ij} \nabla \cdot \mathbf{u}) \) is the viscous stress tensor and

\[ G = \frac{\Delta \rho}{\rho_s} \Delta T, \quad \mathcal{R} = \frac{g \Delta T d^3}{\kappa_s^2 T_s}, \quad Q = \frac{B^2 \alpha^2}{\mu_0 \kappa_s^2}, \quad \sigma = \frac{\mu_s}{\rho_s \kappa_s}, \quad \xi = \frac{\eta_s}{\kappa_s}. \]

In these equations the perturbations \( p_1, \rho_1, T_1, S_1 \) of the basic state have been nondimensionalized using \( \Delta \rho = \rho_s \kappa_s^2/d^2, \Delta \rho, \Delta T, \) and \( c_p \Delta T/T_s \), respectively, implying that

\[ \frac{\Delta \rho}{\Delta T \mathcal{R} \rho_s} = \frac{\delta_T}{\gamma - 1}. \]

In addition, the velocity field and time were scaled using the diffusive scales \( \kappa_s/d \) and \( d^2/\kappa_s \) across the depth \( d \), and the magnetic field using the imposed field \( B_0 \).

These equations are complemented by the linearized equation of state

\[ \frac{\gamma}{\gamma - 1} \frac{\delta_T}{\rho_0} p_1 = G \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0} \]

and the thermodynamic relations

\[ S_1 = \frac{T_1}{T_0} \frac{\delta_T}{\mathcal{R}} \frac{\rho_1}{\rho_0} - G \frac{\rho_1}{\rho_0} + \frac{1}{\gamma - 1} \frac{\delta_T}{\mathcal{R}} \frac{p_1}{\rho_0}. \]

Using Eq. (80), the momentum equation [Eq. (73)] can be expressed in the more convenient form

\[ D_t \mathbf{u} = -\nabla \left( \frac{p_1}{\rho_0} \right) + \mathcal{R} \mathbf{S}_1 \hat{z} + \frac{Q}{\rho_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{\sigma}{\rho_0} \nabla \cdot \tau. \]

**Asymptotic development**

We consider here the case where the fluid is permeated by an imposed uniform vertical magnetic field. The dimensionless magnetic field is thus assumed to be the superposition \( \mathbf{B} = \hat{z} \)}
Thus the fluctuating velocity field + \mathbf{b}, where \mathbf{b} denotes the perturbation from the imposed field. Simplified reduced equations can be obtained in the strong field limit, \( Q \gg 1 \), using a multiple-scale expansion in the vertical direction and time, with

\[
\nabla \rightarrow Q^{1/4} \nabla + \hat{z} \partial_z, \quad \text{Ra} \rightarrow Q \text{Ra}, \quad \partial_t \rightarrow Q^{1/2} \partial_t + \partial_t, \quad (82)
\]

\[
u \rightarrow Q^{1/4} \nu, \quad \mathbf{b} \rightarrow Q^{-1/4} \mathbf{b}, \quad p_1 \rightarrow Qp_1. \quad (83)
\]

All other quantities remain unscaled. We assume that \( \mu_0 \), \( \eta_0 \), and \( K_0 \) as well as the background stratification depend on \( Z \) only.

Under the above rescaling the magnetoanelastic equations become

\[
- Q^{1/4} \nabla \left( \frac{p_1}{\rho_0} \right) + \frac{1}{\rho_0} (\nabla \times \mathbf{b}) \times \hat{z} \left( \partial_z \left( \frac{p_1}{\rho_0} \right) - \frac{\text{Ra} S_1}{\text{Ra}} \right) \hat{z} = Q^{-1/4} \left( \frac{\sigma}{\rho_0} \nabla \cdot \tau - \frac{1}{\rho_0} (\hat{z} \times \partial_z \mathbf{b}) \times \hat{z} - \frac{1}{\rho_0} (\nabla \times \mathbf{b}) \times \mathbf{b} \right) + \text{h.o.t.}, \quad (84)
\]

\[
\nabla \times (\nu \times \hat{z}) = Q^{-1/4} \left( \frac{\eta_0}{\rho_0} \partial_t \mathbf{b} - \nabla \times (\nu \times \mathbf{b}) - \hat{z} \times \partial_z (\nu \times \hat{z}) - \zeta \eta_0 \nabla^2 \mathbf{b} \right) + \text{h.o.t.}, \quad (85)
\]

\[
(\nabla + Q^{-1/4} \hat{z} \partial_z) \cdot (\rho_0 \nu) = 0, \quad (\nabla + Q^{-1/4} \hat{z} \partial_z) \cdot \mathbf{b} = 0, \quad (86)
\]

\[
p_0 T_0 (D_0 S_1 + Q^{-1/4} \nu \partial_z S_1 + Q^{-1/2} \partial_z S_1) = (\nabla + Q^{-1/4} \hat{z} \partial_z) \cdot (K_0 \nabla + Q^{-1/4} \hat{z} \partial_z) T_1
\]

\[
+ Q^{-1/2} \lambda \eta \left( \text{Ra} \right) \frac{\partial_t}{\text{Ra}} \mathbf{b} \cdot \mathbf{b} = Q^{-1/2} \lambda \eta \left( \text{Ra} \right) \frac{\partial_t}{\text{Ra}} \mathbf{b} \cdot \mathbf{b}^2 + Q^{-1/2} \lambda \eta \left( \text{Ra} \right) \frac{\partial_t}{\text{Ra}} \mathbf{b} \cdot \mathbf{b}^2 + \text{h.o.t.} \quad (87)
\]

Here \( D_0 = \partial_t + \nu \cdot \nabla \). We now proceed as in Sec. II A and pose an asymptotic expansion of the form \( O(1) \) in powers of the small parameter \( \epsilon = Q^{-1/4} \) for both mean and fluctuating quantities.

At leading order \( O(Q^{1/4}) \) in the governing momentum equation [Eq. (84)], we obtain \( \nabla (\nu p_0/\rho_0) = 0 \), implying that \( \nu p_0 = 0 \) and hence that \( p_1 = \bar{p}_1 (Z, T) \). At \( O(1) \) the sole nontrivial mean balance is

\[
- \partial_z \left( \frac{\bar{p}_1}{\rho_0} \right) + \frac{\text{Ra}}{\text{Ra}} \bar{S}_1 = 0, \quad (88)
\]

indicating hydrostatic balance for the mean quantities at leading order. The \( O(1) \) fluctuating components of the governing equations [Eqs. (84)-(87)] are

\[
- \frac{1}{\rho_0} \nabla \left( \frac{p_1}{\rho_0} + \hat{z} \cdot \mathbf{b}_0' \right) + \frac{1}{\rho_0} \hat{z} \cdot \nabla \mathbf{b}_0' = - \frac{\text{Ra}}{\text{Ra}} \bar{S}_1' \hat{z}, \quad (89)
\]

\[
\nabla \times \left( \mathbf{u}_0' \times \hat{z} \right) = - \hat{z} \nabla \cdot \mathbf{u}_0' + \hat{z} \cdot \nabla \mathbf{u}_0' = 0, \quad (90)
\]

\[
\nabla \cdot \mathbf{u}_0' = 0, \quad \nabla \cdot \mathbf{b}_0' = 0, \quad (91)
\]

\[
p_0 T_0 D_0 S_1' = K_0 \nabla^2 T_1'. \quad (92)
\]

Thus the fluctuating velocity field \( \mathbf{u}_0' \) is independent of \( z \). In addition, we take \( S_1' = T_1' = 0 \) and use the above equations to conclude that \( p_1' \) is independent of \( z \), and that \( p_1' + \hat{z} \cdot \mathbf{b}_0' = 0 \). Hence fluctuations in the gas pressure are directly related to the vertical component of the magnetic field.
perturbation. This is the magnetic analog of the geostrophic balance that arises in rotationally constrained flows (cf. Sec. II A). It now follows that

$$\frac{\bar{G} \bar{p}_{10}}{\rho_0} = - \left( \frac{\bar{T}_{10}}{T_0} - \frac{\gamma}{1 - \gamma \bar{G}} \frac{\bar{p}_{10}}{\rho_0} \right), \quad (93)$$

$$\bar{S}_{10} = \frac{\bar{T}_{10}}{T_0} - \frac{\gamma \bar{G}}{\bar{G}} \left( \frac{\bar{p}_{10}}{\rho_0} + \frac{1}{\gamma - 1} \frac{\bar{G}}{\rho_0} \right), \quad (94)$$

and, in particular, that $\rho_{10}^* = 0$.

At $O(Q^{-1/4})$ the mean components of Eqs. (84)–(86) are

$$\frac{1}{\rho_0} \left( \bar{z} \times \partial_2 \bar{b}_0 \right) \times \bar{z} - \left( \partial_2 \left( \frac{\bar{p}_{11}}{\rho_0} \right) - \frac{\bar{G}}{\rho_0} \bar{S}_{11} \right) \bar{z} = \mathbf{0}, \quad (95)$$

$$\bar{z} \times \partial_2 (\bar{u}_0 \times \bar{z}) = \mathbf{0}, \quad \partial_2 \left( \rho_0 \bar{z} \cdot \bar{u}_0 \right) = 0, \quad \partial_2 (\bar{z} \cdot \bar{b}_0) = 0, \quad (96)$$

implying that $\bar{u}_0 = \mathbf{0}$ and $\bar{b}_0 = \mathbf{0}$. The vertical component of Eq. (95) indicates that the averaged layer remains in hydrostatic balance to this order.

Dynamic equations for the leading-order variables can be deduced from the fluctuating equations [Eqs. (84)–(87)] at $O(Q^{-1/4})$,

$$- \frac{1}{\rho_0} \nabla \cdot \left( p_{12} + \bar{z} \cdot \mathbf{b}_1' \right) + \frac{1}{\rho_0} \bar{z} \cdot \nabla \mathbf{b}_1 = D_i u_i' - \sigma \frac{\mu_0}{\rho_0} \nabla^2 u_i' - \frac{1}{\rho_0} \left( \bar{z} \times \partial_2 \bar{b}_0 \right) \times \bar{z} - \frac{1}{\rho_0} \left( \nabla \times \mathbf{b}_0 + \mathbf{b}_0 \times \mathbf{b}_0 \right) \nabla \cdot \mathbf{b}_1 + \frac{1}{\rho_0} \bar{z} \times \partial_2 \left( \frac{\bar{p}_{11}}{\rho_0} \right) \bar{z}, \quad (97)$$

$$\nabla \times (u_i' \times \bar{z}) = \frac{\partial \bar{b}_0'}{\partial t} - \nabla \times (u_0 \times \bar{b}_0' \times \bar{z}) - \dot{z} \partial_2 \left( \nabla \times \mathbf{b}_0 \right) - \xi \eta \nabla^2 \mathbf{b}_0', \quad (98)$$

$$\rho_0 \nabla \cdot u_i' + \partial_2 \rho_0 u_i' = 0, \quad \nabla \cdot \mathbf{b}_1' + \partial_2 c_1' = 0, \quad (99)$$

$$\rho_0 T_0 \left( D_i S_{11} + w_0' \bar{G} \bar{S}_{10} \right) = K_0 \nabla^2 T_1', \quad (100)$$

where $D_i = \partial_t + u_i' \nabla \cdot$. On taking $\bar{z}$- and $\bar{z} \cdot \nabla$ of Eqs. (97) and (98), and utilizing Eq. (99), we obtain

$$\frac{1}{\rho_0} \bar{z} \cdot \nabla p_{12} = D_i w_0' - \sigma \frac{\mu_0}{\rho_0} \nabla^2 w_0' - \frac{1}{\rho_0} \bar{b}_0' \cdot \nabla \cdot c_0' + \frac{1}{\rho_0} \left( \frac{\bar{p}_{11}'}{\rho_0} \right) - \bar{G} \bar{S}_{11}, \quad (101)$$

$$\frac{1}{\rho_0} \bar{z} \cdot \nabla (\bar{z} \cdot j_1') = \bar{z} \cdot \nabla \left( D_i u_i' - \sigma \frac{\mu_0}{\rho_0} \nabla^2 u_i' - \frac{1}{\rho_0} \left( \bar{z} \partial_2 \bar{b}_0 \right) \times \bar{z} - \frac{1}{\rho_0} \left( \nabla \times \mathbf{b}_0 + \mathbf{b}_0 \times \mathbf{b}_0 \right) \nabla \cdot \mathbf{b}_1 \right), \quad (102)$$

$$\bar{z} \cdot \nabla w_1' = D_i c_0' - \mathbf{b}_0' \cdot \nabla \cdot w_0' - \xi \eta \nabla^2 c_0' - \frac{1}{\rho_0} \partial_2 \rho_0 u_i', \quad (103)$$
\[ \mathbf{z} \cdot \nabla (\mathbf{z} \cdot \omega_1) = \mathbf{z} \cdot \nabla \left( \frac{\partial \mathbf{b}_0'}{\partial t} - \mathbf{V}_\perp \times (\mathbf{u}_0' \times \mathbf{b}_0') - \mathbf{z} \times \partial_z (\mathbf{u}_0' \times \mathbf{z}) - \zeta \eta_0 \nabla_\perp^2 \mathbf{b}_0' \right), \]  

where we have adopted the notation \( \mathbf{u}_0' = (u_0', v_0', w_0') \), \( \mathbf{b}_0' = (b_0', c_0') \); moreover, \( \mathbf{j}_1 = \nabla \times \mathbf{b}_1' \) and \( \omega_1 = \nabla \times \mathbf{u}_1' \). The solvability conditions for these equations follow on averaging over the small-scale variable \( z \) using the results \( \mathbf{z} \cdot \nabla S_{11}' = \mathbf{z} \cdot \nabla \mathbf{T}_{11}' = 0 \) that follow from the thermodynamic relation

\[ S_{11}' = \frac{T_{11}'}{T_0} - \frac{\delta T}{\mathcal{R} a} p_0', \]

in conjunction with \( \mathbf{z} \cdot \nabla \) of Eq. (100) and the fact that both \( w_0' \) and \( p_1' \) are independent of \( z \). We obtain

\[ D_j^r w_0' = \frac{\partial}{\partial t} \left( \frac{c_0'}{\rho_0} \right) + \frac{\mathcal{R} a S_{11}'}{\rho_0} + \frac{1}{\rho_0} \mathbf{b}_0' \cdot \nabla c_0' + \frac{\sigma \mu_0}{\rho_0} \nabla_\perp^2 w_0', \]

\[ \mathbf{z} \cdot \nabla \left( D_j^r \mathbf{u}_0' - \frac{\sigma \mu_0}{\rho_0} \nabla_\perp^2 \mathbf{u}_0' - \frac{1}{\rho_0} (\mathbf{z} \times \partial_z \mathbf{b}_0') \times \mathbf{z} - \frac{1}{\rho_0} (\nabla_\perp \times \mathbf{b}_0') \times \mathbf{b}_0' \right) = 0, \]

\[ D_j^r c_0' = \frac{1}{\rho_0} \partial_z (p_0 w_0') + \mathbf{b}_0' \cdot \nabla w_0' + \zeta \eta_0 \nabla_\perp^2 c_0', \]

\[ \mathbf{z} \cdot \nabla \left( \frac{\partial \mathbf{b}_0'}{\partial t} - \mathbf{V}_\perp \times (\mathbf{u}_0' \times \mathbf{b}_0') - \mathbf{z} \times \partial_z (\mathbf{u}_0' \times \mathbf{z}) - \zeta \eta_0 \nabla_\perp^2 \mathbf{b}_0' \right) = 0, \]

where we have used the result \( p_1' + c_0' = 0 \) to eliminate \( p_1' \) in favor of \( c_0' \). These equations are complemented by Eq. (100) written in the form

\[ D_j S_{11}' + w_0' \partial_z \tilde{S}_{10} = \frac{K_0}{\rho_0} \nabla^2 \left( S_{11}' - \frac{\delta T}{\mathcal{R} a} c_0' \right), \]

and the mean of the energy equation at \( \mathcal{O}(Q^{-1/2}) \) written in the form

\[ \rho_0 a_0' \left( \partial_z \tilde{S}_{10} + \frac{1}{\rho_0} \partial_z (p_0 w_0' S_{11}') \right) = \tilde{\zeta} \mathcal{R} a \left( K_0 \partial_z S_{10} \right) + \tilde{\zeta} \frac{\delta T}{\mathcal{R} a} \partial_z \tilde{V}_\perp \times \tilde{b}_0' + \sigma \delta \tilde{S}_0^2 \tilde{V}_\perp^2, \]  

Equations (105)–(111) form a closed reduced system. The properties of this system have not been studied.

### C. Magnetorotational instability

In the previous sections we have discussed relatively straightforward applications of our approach. In this section we turn to a problem that is of great importance in astrophysics and is currently being studied in the laboratory: the MRI. This is an instability that is driven by differential rotation in the presence of a (typically vertical) magnetic field and occurs even when the angular momentum of the material increases away from the rotation axis (i.e., when the criterion for stability in the absence of the field is satisfied). In these circumstances the inclusion of even weak magnetic field destabilizes the flow and leads to angular momentum transport that is much more efficient than that due to (microscopic) viscosity. As a result the MRI is believed to be responsible for accretion in astrophysical accretion disks, and has been the subject of numerous numerical simulations, and increasingly experiments. However, substantial progress can be made analytically, as demonstrated here in a simple model of the instability relevant to the
laboratory experiments. The theory demonstrates that the saturation process of the instability requires both viscous and Ohmic dissipation, and describes the approach from small amplitude perturbations to the final strongly nonlinear saturated state.

1. Formulation of a model problem

We consider a straight channel \(-L/2 \leq x \leq L/2\), \(-\infty < y < \infty\), \(-\infty < z < \infty\), filled with an electrically conducting incompressible fluid, and rotating about the \(z\) axis with constant angular velocity \(\Omega\). We suppose that a linear shear flow \(U_0=(0,\alpha x,0)\), \(\alpha < 0\), is maintained in the channel, for example, by boundaries that slide in the \(y\) direction with speeds \(\pm \sigma L/2\). In addition, we suppose that a constant magnetic field \(B_0=(0,B_{\text{pol}},B_{\text{pol}})\) is present and consider \(y\)-independent perturbations of this state. These can be written in the form \(u = (u, v) = (-\psi_z, v, \phi_r), b = (a, b, c) = (-\phi_z, b, \phi_r)\) and satisfy the dimensionless equations

\[
\nabla^2 \psi_z + 2\psi_z \nabla^2 \psi = v_A^2 \nabla^2 \phi_z + v_A^2 J(\phi, \nabla^2 \phi) + \nu \nabla^4 \psi, \tag{112}
\]

\[
v_z - (2\Omega + \sigma) \psi_z + J(\psi, v) = v_A^2 b_z + v_A^2 J(\phi, b) + \nu \nabla^2 v, \tag{113}
\]

\[
\phi_z + J(\psi, \phi) = \psi_z + \eta \nabla^2 \phi, \tag{114}
\]

\[
b_z + J(\psi, b) = v_z - \sigma \phi_z + J(\phi, v) + \eta \nabla^2 b, \tag{115}
\]

where \(v_A = B_{\text{pol}}/\sqrt{\mu_0 \sigma} U\) is proportional to the Alfvén speed associated with the vertical magnetic field, \(J(\psi, \phi) = f_{x,y} - f_{x,y}, \sigma\), and \(\Omega, v, \eta\) represent the dimensionless rotation rate, kinematic viscosity, and Ohmic diffusivity, nondimensionalized using a velocity scale \(U\) and the channel width \(L\). Note that \(B_{\text{rot}}\) drops out of these equations. This is not so in an annulus, where hoop stresses are present, and the MRI becomes oscillatory.52,53

In the following we assume that the MRI takes the form of thin fingers propagating in the \(x\) direction, as indicated by linear stability theory54 and subsequent numerical experiments.55,56 By definition, such fingers have a small transverse width. Accordingly we introduce a small parameter \(\epsilon \ll 1\) and suppose that all derivatives in the \(z\) direction are large, i.e., we scale \(z\) such that in the new variable \(\hat{z}\) is replaced by \(\epsilon^{-1} \hat{z}\). In addition we suppose that in an appropriate dimensionless sense the dissipative processes are weak, and let \((\nu, \eta) = \epsilon (\hat{\nu}, \hat{\eta})\). At the same time we assume that the system is rotating rapidly and that the dimensionless shear rate is also large, i.e., we set \((\Omega, \sigma) = \epsilon^{-1} (\hat{\Omega}, \hat{\sigma})\) while keeping \(v_A\) of order unity. These assumptions reflect the conditions generally believed to be present in accretion disks: the shear is the dominant source of energy for the instability, but the instability itself requires the presence of a (weaker) vertical magnetic field. Dissipative effects are weaker still but cannot be ignored since they are ultimately responsible for the saturation of the instability.

It is important to specify the meaning of the small parameter \(\epsilon\). To do so we select the velocity scale \(U\) by the requirement that \(v_A = 1\). It follows that \(\hat{\Omega} \gg 1\) requires \(\Omega L \gg v_A^*, \eta / v_A^*\) denotes dimensional quantities, while \(\sigma / v_A^*\) requires \(|\sigma| L^2 \gg v_A^*\). In addition, the requirements \(\eta \ll 1\), \(\nu \ll 1\) are equivalent to \(\eta < v_A^*, \nu < v_A^* L^{-1}\). In our scaling these inequalities are related by the requirement that the Elsasser number \(\Lambda = v_A^2 / \Omega \eta = O(1)\), this being a regime of particular interest.57 Moreover, in the asymptotic regime both the Lundquist number \(S = B_{\text{pol}}^* L / \eta / \mu_0 \sigma\) and the magnetic Reynolds number \(Rm = |\sigma|^2 L^2 / \eta^*\) are large and satisfy \(Rm \sim S^2\). This relation falls in the middle of the parameter range where a nonaxisymmetric \((m = 1)\) MRI mode has been observed in a laboratory experiment (Fig. 13).

In parallel with the above assumptions, we need to make further assumptions about the relative magnitude of the various fields. The rapid shearing by the azimuthal flow suggests that we take \((\psi, \phi) \rightarrow \epsilon (\hat{\psi}, \hat{\phi}), (\nu, b) \rightarrow \epsilon^{-1} (\hat{\nu}, \hat{b})\). It turns out that this assumption leads to a self-consistent fully nonlinear stationary solution, satisfying the scaled equations...
2\epsilon^2\hat{\Omega}v_z + \epsilon J(\psi, (\hat{\alpha}^2 + \epsilon^{-2}\hat{\beta}^2))\psi = v_A^2(\hat{\alpha}^2 + \epsilon^{-2}\hat{\beta}^2)\phi_z + \epsilon v_A^2 J(\phi_z, (\hat{\alpha}^2 + \epsilon^{-2}\hat{\beta}^2)) + \epsilon^2(\hat{\alpha}^2 + \epsilon^{-2}\hat{\beta}^2)^2\psi, 

(116)

-\epsilon^{-1}(2\hat{\Omega} + \hat{\varphi})\psi_z + \epsilon^{-1}J(\psi, v) = \epsilon^{-2}v_A^2\hat{b}_z + \epsilon^{-1}v_A^2 J(\phi, b) + \hat{\nu}(\hat{\alpha}^2 + \epsilon^{-2}\hat{\beta}^2)v, 

(117)

\epsilon J(\psi, \phi) = \psi_z + \epsilon^2\hat{\nu}(\hat{\alpha}^2 + \epsilon^{-2}\hat{\beta}^2)\phi_z, 

(118)

\epsilon^{-1}J(\psi, b) = \epsilon^{-2}v_z - \epsilon^{-1}\hat{\nu}\phi_z + \epsilon^{-1}J(\phi, v) + \hat{\nu}(\hat{\alpha}^2 + \epsilon^{-2}\hat{\beta}^2)b. 

(119)

In these equations we have retained the parameter $v_A$ in order to be able to vary the strength of the poloidal field independently of other dimensionless quantities.

2. Results

To solve Eqs. (116)–(119), we posit an expansion of the form $\psi(x, z) = \psi_0(x, z) + \epsilon\psi_1(x, z) + \cdots$ with similar expressions for the other three fields and look for structures that are periodic in the $z$ direction. From Eqs. (117) and (119) it now follows that at $O(\epsilon^{-2}),$

$$v_A^2\hat{b}_{0z} + \hat{\nu}v_{0zz} = 0, \quad v_{0z} + \hat{\nu}b_{0zz} = 0,$$

(120)

and hence that

$$v_0 = V(x), \quad b_0 = B(x),$$

(121)

while from Eq. (118) we obtain at $O(1),$
Finally, Eqs. (117) and (119) imply

\[-(2\hat{\Omega} + \hat{\sigma})\psi_{0z} + J(\psi_{0z},v_{0}) = v_{A}^{2}b_{1z} + v_{A}^{2}J(\phi_{0},b_{0}) + \hat{\psi}v_{1zz},\]  

(123)

\[J(\psi_{0z},b_{0}) = v_{1z} - \hat{\sigma}\phi_{0z} + J(\phi_{0},v_{0}) + \hat{\psi}b_{1z}.\]  

(124)

In the following we write

\[\psi_{0} = \frac{1}{2}(\Psi(x)e^{inz} + \text{c.c.}), \quad v_{1} = \frac{1}{2}(\mathcal{V}(x)e^{inz} + \text{c.c.}),\]  

(125)

\[\phi_{0} = \frac{1}{2}(\mathcal{F}(x)e^{inz} + \text{c.c.}), \quad b_{1} = \frac{1}{2}(\mathcal{B}(x)e^{inz} + \text{c.c.}).\]  

(126)

From Eqs. (122)–(124) it now follows that

\[\mathcal{F} = \frac{i\Psi}{\hat{\eta}m},\]  

(127)

\[\mathcal{V} = \frac{(v_{A}^{2} + \hat{\psi}^{2}n^{2})\mathcal{V}^{0} + \hat{\psi}^{2}n^{2}(2\hat{\Omega} + \hat{\sigma}) + v_{A}^{2}\hat{\sigma} + i\Psi}{n\hat{\sigma}(v_{A}^{2} + \hat{\psi}n^{2})},\]  

(128)

\[\mathcal{B} = \frac{i(v_{A}^{2} + \hat{\psi}n^{2})B^{0} + n\hat{\sigma}(\hat{\sigma} + \mathcal{V}) - \hat{\psi}(2\hat{\Omega} + \hat{\sigma} + \mathcal{V}^{0})}{n\hat{\sigma}(v_{A}^{2} + \hat{\psi}n^{2})}.\]  

Finally, Eqs. (117) and (119) yield at \(O(1)\) the results

\[-(2\hat{\Omega} + \hat{\sigma})\psi_{1z} + J(\psi_{0z},v_{1}) + J(\psi_{1z},v_{0}) = v_{A}^{2}b_{2z} + v_{A}^{2}J(\phi_{0},b_{1}) + v_{A}^{2}J(\phi_{1},b_{0}) + \hat{\psi}(v_{0x} + v_{2zz}),\]  

(129)

and

\[J(\psi_{0z},b_{1}) + J(\psi_{1z},b_{0}) = v_{2z} - \hat{\sigma}\phi_{1z} + J(\phi_{0},v_{1}) + J(\phi_{1},v_{0}) + \hat{\psi}(b_{0x} + b_{2zz}).\]  

(130)

Averaging these equations over \(z\) and using the fact that the quantities \(\psi_{0}, \phi_{0}, v_{0}, v_{1}, b_{0}, b_{1}\) are all, by construction, periodic in \(z\) with zero mean yields the following pair of equations:

\[\hat{\psi}V'' = \hat{\sigma}_{z}(\phi_{0}v_{1z}) - v_{A}^{2}\hat{\sigma}_{z}(\phi_{0}b_{1z}),\]  

(131)

\[\hat{\phi}B'' = \hat{\sigma}_{z}(\phi_{0}b_{1z}) - \hat{\sigma}_{z}(\phi_{0}v_{1z}).\]  

(132)

Thus in the bulk, away from the boundaries at \(x = \pm 1/2\), \(V''\) and \(B''\) satisfy

\[\hat{\psi}V'' = \psi_{0}v_{1z} - v_{A}^{2}\phi_{0}b_{1z},\]  

(133)

\[\hat{\phi}B'' = \phi_{0}b_{1z} - \phi_{0}v_{1z} + C,\]  

(134)

where the constant \(C\) is determined by force balance across the channel in the saturated state. Evaluating the averages in terms of \(\Psi\) and solving for \(V''\) and \(B''\) now yields
\[ V'(x) = -\frac{(1/2)\beta |\Psi|^2}{\hat{\nu} + (1/2)\alpha |\Psi|^2}, \]  
Equation (135) determines the equilibrated shear \( V' \) in terms of the stream function amplitude \( |\Psi| \) and wave number \( n \) of the instability. The result does not depend on the choice of \( C \).

We can obtain an additional relation between \( |\Psi| \) and \( n \) from Eq. (116). This equation yields at leading order

\[ 2\hat{\Omega}v_{1z} = v_{zzz}^2 + \hat{\nu}\psi_{zzzz} \]  
or, equivalently,

\[ 2\hat{\Omega}(v_{zzz}^2 + \hat{n}^2)\hat{\nu}V'' + (2\hat{\Omega} + \hat{\sigma})\hat{n}^2(n^2 + \hat{\nu}^2v_{zzz}^2) = 0. \]  
Equation (140) determines the amplitude \( |\Psi| \) of the equilibrated state in the bulk as a function of the imposed shear rate \( \hat{\sigma} < 0 \) for each choice of wave number \( n \) and therefore represents the bifurcation equation for this problem. Moreover, it demonstrates that \( \Psi, V', \) and \( B' \) are all independent of the radial coordinate \( x \).

The theory thus far remains unsatisfactory in one respect: the wave number \( n \) is not specified. It is usual in these circumstances to use the wave number \( n_{\text{max}} \) of the fastest growing mode. According to linear theory the growth rate \( \lambda \) is given by

\[ 2\hat{\Omega}(2\hat{\Omega} + \hat{\sigma})(\lambda + \hat{n}^2v_{zzz}^2 + \hat{\nu}^2v_{zzzz}^2) + [(\lambda + \hat{n}^2)(\lambda + \hat{n}^2v_{zzz}^2 + \hat{\nu}^2v_{zzzz}^2)^2 = 0, \]  
and \( n_{\text{max}} \) is defined by \( d\lambda/dn=0 \). In Fig. 14 we show the result of using this wave number to compute the saturated state shear \( V'_{\text{max}} \) and the associated stream function amplitude \( |\Psi|_{\text{max}} \) as a function of \( \hat{\eta} \) when \( \Omega = -2\hat{\sigma}/3, v_A = 1, \hat{\nu} = 10^{-6} \); for comparison \( \hat{\sigma} = -1.5 \) corresponds to Kepler shear. Note that comparison with Eq. (140) indicates that in our asymptotic regime the MRI instability is quenched to zero growth rate by the change in the shear rate that it produces.

### 3. Reduced equations

It is of interest to examine the approach from an initial small amplitude state with \( V' = 0 \) to the final equilibrated state [Eq. (135)]. To do this we note that the MRI evolves initially on a
dynamical or rotation time scale, i.e., the fast time \( t/H_9280 = O(1) \). However, examining the structure of the equations in the long time limit, we also notice that the final approach to the saturated state proceeds on the much slower time scale \( T/H_9280 = O(1) \), i.e., on the resistive time scale. These observations suggest a multiple time scale expansion, but we have not succeeded in making such an expansion systematic. Instead, we use ansatz (125) to reduce the equations

\[
\psi_{0zz\tau} + 2\dot{\Omega}b_{1z} = \nu_A^2 \phi_{0zzzz} + \dot{\nu}\psi_{0zzzzz},
\]

\[
v_{1\tau} - (2\dot{\Omega} + \dot{\sigma} + V')\phi_{0z} = \nu_A^2 b_{1z} - \nu_A^2 B' \phi_{0z} + \dot{\nu} v_{1zz},
\]

\[
\phi_{0\tau} = \psi_{0z} + \dot{\psi} \phi_{0zz},
\]

\[
b_{1\tau} - \psi_{0z} B' = v_{1z} - (\dot{\sigma} + V') \phi_{0z} + \dot{\psi} b_{1zz}
\]

to a set of ordinary differential equations

\[
\Psi_{\tau} + \dot{\psi} n^2 \Psi - 2i n^{-1} \dot{\Omega} \nu - inv_A^2 \mathcal{F} = 0,
\]

\[
\n_{\tau} + \dot{\psi} n^2 \nu = in(2\dot{\Omega} + \dot{\sigma} + V')\Psi + inv_A^2 B - inv_A^2 B' \mathcal{F},
\]

\[
\mathcal{F}_{\tau} + \dot{\psi} n^2 \mathcal{F} = in\Psi,
\]

\[
\mathcal{B}_{\tau} + \dot{\psi} n^2 \mathcal{B} = in\Psi B' + in \nu - in(\dot{\sigma} + V') \mathcal{F},
\]

and adopt relations (133) and (134) for \( V' \) and \( B' \),

\[
\dot{\psi} V' = -\frac{1}{4} in(\Psi^*/\Psi) - \Psi^* \nu + \frac{1}{4} inv_A^2 (\mathcal{F} B^* - \mathcal{F}' B).
\]
Beyond a brief transient these equations appear to capture correctly the saturation process and, in particular, the final steady state. The resulting evolution of $V'$ and the associated stream function amplitude $|\Psi|$ are shown in Fig. 15 starting from small amplitude initial conditions, assuming that relations (151) and (152) hold at each instant in time. The evolution is shown over times $\tau = O(\epsilon^{-2})$ and shows unambiguously the convergence of $V'$ toward its saturated value [Eq. (135)].

It should be mentioned that a similar reduction is possible in polar coordinates as well, even though in the presence of a toroidal field the primary instability is now an oscillatory instability instead of a steady state one. The curvature of the flow and the toroidal field now distinguishes between the inward and outward directions, raising the possibility that our nonlinear results can be used to compute self-consistently the radial transport of angular momentum. Such results may well be applicable to recent experiments\(^{50}\) and simulations\(^{58}\) of the MRI in the Taylor-Couette geometry. The latter bear substantial similarity to the predictions described above, although the wavelength of the instability in the nonlinear regime is larger than that predicted by the linear theory result [Eq. (142)].

**D. Geophysical flows**

In many cases the approach described here permits the self-consistent computation of mean flows and heat fluxes.\(^{39}\) An example is provided by convection on an $f$ plane, i.e., convection in a plane layer at latitude $\Lambda$, $0 < \Lambda < \pi/2$, with horizontal scales sufficiently small that the variation of the local rotation rate with latitude can be ignored. In the rapid rotation limit this problem also leads to a reduced description; moreover, the leading-order single mode solutions are independent of the mean flow and heat flux present due to the tilt of the convection cells relative to the local vertical and can therefore be used to compute these mean quantities self-consistently in the fully nonlinear regime. A number of general conclusions about the magnitude and the direction of these flows follow from this analysis, some of which are independent of the details of the single mode solution. It is significant that these conclusions agree with three-dimensional DNS of this system carried out at $O(1)$ Rossby numbers.\(^{60}\)

Related computations can be performed for convection in an inclined magnetic field. Here, too, it is possible to compute self-consistently the mean flows that result from the tilt of the convection cells in the field.\(^{40}\)

Other flows where the present approach may be applied include the salt-finger problem (i.e., the diffusive instability of a hot salty liquid above a colder fresher liquid\(^{61}\)) and the mathematically analogous Goldreich-Schubert-Fricke instability that arises when angular momentum decreases
outward along surfaces of constant gravitational potential in a differentially rotating star.\(^{62,63}\) In both these problems the single mode problem is degenerate, but the reduced equations may well capture correctly the dynamics resulting from the instability.

V. SUMMARY AND OUTLOOK

In this review we have described a new set of tools that have the potential to reach the extreme parameter regimes that are characteristic of almost all geophysical and astrophysical flows. These flows are at present not accessible to fully resolved three-dimensional simulations and are unlikely to become so in the near future. The techniques described here lead to systems of reduced or simplified equations with a well defined regime of validity. These equations are easier to simulate since the fast time scales and small spatial scales associated with the large parameter are filtered out. As a result the reduced equations are valid in the bulk, outside of (typically passive) boundary layers. We have illustrated the approach with a detailed derivation of the reduced equations for rapidly rotating Rayleigh-Bénard convection, together with the solution to these equations at highly supercritical Rayleigh numbers. These have revealed a number of new phenomena that appear to be characteristic of low Rossby number convection but have hitherto not been studied because of the inaccessibility of this regime. Moreover, we have pointed out that the reduced equations possess an important class of nonlinear single mode solutions that can be obtained semianalytically by solving a nonlinear eigenvalue problem for, typically, a flux or a torque. In convection problems the associated eigenfunction describes the approach to an isothermal interior as the Rayleigh number increases to higher and higher values; in differentially rotating systems it may describe the change in the background shear due to an instability. The technique applies to nonlinear oscillations as well, and as we have seen, to the saturation of other instabilities, such as the magnetorotational instability, far from threshold. The approach resembles the single mode approach originally put forward by Spiegel and co-workers\(^{64-66}\) but is both systematic and asymptotically exact.

It remains to consider the stability of the single mode solutions. We have seen that these may be stable (as in the MRI) or unstable (as in rapidly rotating convection). The difference between these two problems can be found in the spatial scales admitted by the theory. The single mode solutions are typically stable with respect to perturbations with the same scale, unless one includes competing patterns with the same scale; the resulting pattern competition is determined by eigenvalues that become nonzero only at higher order. Instability to a mode that tilts the structure and generates a mean shear may also be present.\(^{45}\) However, the integration of the reduced system for rapidly rotating convection\(^{34}\) suggests that these spatially periodic structures typically become susceptible to large-scale modes that lead to disorder and spatiotemporal dynamics. The amplitude at which these modes set in, and the properties of the resulting chaotic state have not hitherto been studied in a systematic way. It is clear, however, that the reduced equations are capable of describing a turbulence-like state, albeit with a reduced number of degrees of freedom.

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