DOUBLY DIFFUSIVE WAVES

BY

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ABSTRACT. The theory of the degenerate Hopf bifurcation with $O(2)$ symmetry is applied to doubly diffusive convection with periodic boundary conditions in the horizontal. The theory predicts that near the Hopf bifurcation only travelling waves can be stable. This result is in agreement with numerical solutions of the partial differential equations.

1. Introduction. Over the past few years it has been realized that systems with symmetries can exhibit a rich variety of bifurcation phenomena. Such bifurcations are often nongeneric—they do not occur generically in systems without symmetry—but they may be generic, and therefore readily observed, in systems possessing a particular symmetry. For this reason bifurcation with symmetry has been the focus of much recent research. The role played by the symmetry has two facets. On the one hand, the symmetry is responsible for the generic appearance of multiple eigenvalues. On the other hand, it facilitates the study of bifurcations at such multiple eigenvalues by restricting the structure of the normal form equations that describe them, and their unfoldings, and so makes much of the new and interesting secondary and tertiary bifurcation behavior more accessible than it is in nongeneric multiple bifurcation problems.

We illustrate these ideas using a particular system that has been much studied by fluid dynamicists, but where the role played by the symmetry has not until recently been recognized. The system, thermosolutal convection, is typical of all doubly diffusive systems which include magnetoconvection [1], convection in a rotating layer [2] and convection in binary mixtures [3]. Consider a layer of fluid confined between infinite horizontal planes and heated from below. If the fluid contains a stabilizing solute gradient (e.g., salt), then
the initial instability of the motion-free state may be a Hopf bifurcation. To understand the physical reason for this, and highlight the role played by the solute gradient, consider an upward displacement of a fluid element. Such a fluid element finds itself in a cooler, fresher environment, and rapidly cools while maintaining its solute content. It therefore acquires negative buoyancy, and begins to descend. At its original position it is still too dense because it is now cooler, and hence it overshoots by more than its upward displacement. In this way growing oscillations set in. Evidently, the instability requires that \( r \equiv \kappa_S/\kappa_T < 1 \) where \( \kappa_S \) and \( \kappa_T \) are, respectively, the solutal and thermal diffusivities, and that the restoring force produced by the solute "spring" is large enough to overcome viscous drag.

In this paper we shall be concerned with the nonlinear equilibration of these oscillations as a function of the bifurcation parameter \( \lambda \propto R_T - R_T^2 \), where \( R_T \) is the thermal Rayleigh number. We shall confine attention to two spatial dimensions, involving the horizontal variable \( x \) and the vertical variable \( z \), and time \( t \), and consider spatially periodic solutions with periodicity \( d \) in the horizontal. The system is then invariant under horizontal translations through a distance \( \ell \, (\text{mod} \, d) \), and under reflection in the line \( x=0 \). Thus if \( \Psi(x, z, t) \) is the streamfunction, then \( \Psi(x + \ell, z, t) \) is also a streamfunction, and so is \( -\Psi(-x, z, t) \). We call the generation of new solutions from old ones a symmetry of the system. In the present case we have the symmetries

\[
\begin{align*}
(1a) \quad \text{translation} : & \quad \Psi(x, z, t) \rightarrow \Psi(x + \ell, z, t) \\
(1b) \quad \text{reflection} : & \quad \Psi(x, z, t) \rightarrow -\Psi(-x, z, t).
\end{align*}
\]

The translation symmetry may be identified with the group \( SO(2) \) of rotations of a circle, and the reflections with reflections in the origin \( Z_2 \). The symmetry of the problem is the semi-direct product of these groups, i.e., the group \( O(2) \). Owing to the presence of the \( O(2) \) symmetry the eigenvalues \( \pm i\omega_0 \) at the Hopf bifurcation are doubled, and we expect multiple solution branches to bifurcate from \( \lambda = 0 \). We shall show that these are the branches of standing waves (SW) and travelling waves (TW). The standing waves are invariant under the reflection \( Z_2 \), but not under the translations \( SO(2) \); the travelling waves propagate either to the left or to the right and hence break the \( Z_2 \) symmetry, but are invariant under the translation \( SO(2) \) coupled with an appropriate (temporal) phase shift.

In the following section we describe our numerical results on thermosolutal convection. In §3 we provide a theoretical interpretation of these results, and in particular show
that near the bifurcation only the travelling waves are stable. In §4 we mention extensions
and applications of our study.

2. Numerical Results. Thermosolutal convection in two dimensions is described by the
non–dimensionalized equations [4]

\[ \frac{1}{\sigma} [\nabla^2 \Psi_t + J(\Psi, \nabla^2 \Psi)] = R_T \Theta_x - R_S \Sigma_x + \nabla^4 \Psi \]
\[ \Theta_t + J(\Psi, \Theta) = \Psi_x + \nabla^2 \Theta \]
\[ \Sigma_t + J(\Psi, \Sigma) = \Psi_x + \tau \nabla^2 \Sigma, \]

where \( \Psi \) is the streamfunction, \( \Theta \) and \( \Sigma \) represent the departures of the temperature and
solute concentration from their conduction profiles and \( J(f, g) \equiv f_x g_z - f_z g_x \). The equations contain four dimensionless parameters: the thermal and solutal Rayleigh numbers
\( R_T \) and \( R_S \), and the Prandtl numbers \( \sigma \equiv \nu/\kappa_T \) and \( \tau \). Here \( \nu \) is the kinematic viscosity.
The parameter \( R_T \) measures the thermal forcing of the system while \( R_S \) measures the
restoring force of the solute gradient. The boundary conditions we employ are:

\[ \Psi = \Psi_{xz} = \Theta = \Sigma = 0 \quad \text{on} \quad z = 0, 1. \]

These describe stress–free boundaries at the top and bottom, with fixed temperature and
solute concentration. We seek solutions that are periodic in the horizontal with period
\( d = 2\pi/k \), where \( k \) is the wavenumber of the mode that first sets in as \( R_T \) is increased; linear
theory shows that \( k = \pi/\sqrt{2} \). However, for convenience in defining the computational mesh
we choose the periodicity to be \( d = 3 \). Most of our nonlinear numerical simulations are
carried out with \( R_S = 10^4, \sigma = 1, \tau = 10^{-1/2}(= 0.31623) \); these values are chosen in order
to be able to resolve easily the thermal and solutal boundaries at the top and bottom of the
layer and to make contact with earlier studies (e.g. [5, 4]). With these parameter values
the Hopf bifurcation occurs at \( R_T^2 = 7.7 \times 10^3 \). The calculations utilize finite difference
schemes in space and time, and typically employ a 66 \times 33 \) spatial grid. Two different
numerical codes have yielded nearly indistinguishable results for the parameters that we
have surveyed.

In Fig. 1 we show a left–travelling wave at \( R_T = 9.1 \times 10^3 \) at three successive times,
after the transients have died away. The times are in units of the thermal diffusion time
in the vertical. The figure shows the velocity field and lines of constant temperature and
FIGURE 1. Left-travelling wave for $\sigma = 1$, $\tau = 10^{-1/2}$, $R_T = 9.1 \times 10^9$, $R_S = 10^4$ and $d = 3$. The velocity field and contours of constant temperature and solute concentration are shown on $x-z$ coordinates at three successive times (in units of the vertical diffusion time). The diagonal dashed line serves to emphasize the lateral displacement of the patterns with time.

solute concentration in the $x-z$ plane. The velocity field is indicated by sets of arrows where the length of an arrow is proportional to the velocity at its midpoint. These arrows follow the streamlines. The gyres can be seen to be translating uniformly to the left, so that in a co-moving frame the pattern is steady. However, at a fixed point $x$, the vertical velocity reverses as successive gyres pass by, as shown in Fig. 2a.

The heat transported by the fluid layer is measured by the dimensionless Nusselt number

$$N(z,t) = 1 + \frac{1}{d} \int_0^d dx \left(-\Theta_z + \Psi_z \Theta\right).$$

The quantity $N - 1$ is proportional to the square of the amplitude of motion; for travelling waves it is independent of both $z$ and $t$. This is shown in Fig. 2b, where $N$ evaluated at a specific $z$ is displayed as a function of $t$. Note the decaying transient.
FIGURE 2. As in Fig. 1, but showing as functions of time (a) the vertical velocity sampled at a particular point in space, and (b) the Nusselt number $N$.

FIGURE 3. For $\sigma = 1$, $\tau = 10^{-1/2}$, $R_S = 10^4$ and $d = 3$ (a) the Nusselt number, and (b) the travelling speed $c$, as functions of the Rayleigh number $R_T$. The arrow indicates the Hopf bifurcation.
FIGURE 4. The instability of a standing wave to a travelling wave, for $\sigma = 1$, $\tau = 10^{-1/2}$, $R_T = 9.5 \times 10^3$, $R_S = 10^4$, $d = 3$. At time $t = 17$ the boundary condition was changed from no-horizontal-flux to periodic. Note that the Nusselt number for the travelling wave is higher than the time-averaged Nusselt number for the standing wave which is 1.665.

In Figs. 3a, b we show the Nusselt number and the travelling velocity $c$ as a function of $R_T$ for steadily travelling waves. We see that both bifurcate from $R_T^2$ at which $N = 1$ and $c = \omega_b / k$, and that the travelling velocity decreases monotonically as the amplitude increases. This behavior is typical of nonlinear oscillators. A more comprehensive description of these results is given in [6].

Numerical results show that with periodic boundary conditions standing waves are unstable to travelling waves. Standing waves can, however, be computed by imposing an additional boundary condition in the horizontal:

$$\Theta_x = \Sigma_x = 0 \quad \text{on} \quad x = 0, d.$$

These conditions prevent horizontal fluxes, and hence travelling waves without affecting the standing waves, and thereby enable us to compute unstable standing waves using the same numerical procedure. Standing waves have been investigated in detail [4, 7, 8].

We show in Fig. 4 what happens to a standing wave when the boundary conditions are changed from (5) to periodic. This calculation is a typical example displaying $N$ as the standing wave decays into a travelling wave, either left- or right-travelling, depending
on initial conditions. Here the standing wave is an asymmetrical oscillation for which the “up” portion of the time trace differs from the “down”.

3. Theory. In this section we describe the theory that is relevant to the results just described. We make full use of the symmetries (1) in the analysis. At $R_T^o$ we write the streamfunction $\Psi$ in the form

$$\Psi(x, z, t) = [(v + w)e^{ikz} + (\bar{v} + \bar{w})e^{-ikz}] \sin \pi z,$$

where

$$\dot{v} = i\omega_v v, \quad \dot{w} = -i\omega_w w.$$

The complex amplitudes $v(t)$ and $w(t)$ are the amplitudes of left- and right-travelling waves. A standing wave results when $v = \bar{w}$. We seek to construct nonlinear amplitude equations for $v, w$ as a function of the bifurcation parameter $\lambda \propto R_T - R_T^o$. The form of these equations is dictated by the symmetries (1), which induce the following action of the group $O(2)$:

$$\begin{align*}
(8a) \quad & \text{translation : } (v, w) \to e^{ikl}(v, w) \\
(8b) \quad & \text{reflection : } (v, w) \to -(\bar{w}, \bar{v}).
\end{align*}$$

There are three invariants of this action,

$$\sigma_1 = |v|^2 + |w|^2, \quad \sigma_2 = v\bar{w}, \quad \sigma_3 = \bar{v}w,$$

and the equivariant vector fields are generated by $(v, 0), (w, 0), (0, v),$ and $(0, w)$. Thus the most general equivariant vector field is of the form

$$\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ \bar{g}_2 & \bar{g}_1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix},$$

where $g_i = g_i(\lambda, \sigma_1, \sigma_2, \sigma_3), i = 1, 2$. We next expand the invariant functions $g_i$ in a Taylor series about the origin, and apply near-identity coordinate changes to remove as many of
the terms as possible. We find that in the new coordinates equation (10) can be written in the form (cf. [9]):

\begin{align}
\dot{v} &= g(\lambda, \sigma_1, |w|^2)v \\
\dot{w} &= \bar{g}(\lambda, \sigma_1, |v|^2)w.
\end{align}

We can do this because in a Hopf bifurcation the invariance under a phase shift in time introduces an additional \(S^1\) symmetry. This is the symmetry

\[ (v, w) \rightarrow (e^{i\theta} v, e^{-i\theta} w). \]

The functions \(\sigma_2, \sigma_3\) are not invariant under \(O(2) \times S^1\). However, by appropriately coupling spatial translations and phase shifts, we see that there is a new symmetry

\[ (v, w) \rightarrow (e^{2i\theta} v, w), \quad (v, w) \rightarrow (v, e^{-2i\theta} w), \]

whose invariants are the functions \(|v|^2, |w|^2\). The result (11) is a consequence of this symmetry. Although equations (11) follow from (10) to all orders in \(v, w\), the \(S^1\) symmetry is not an exact symmetry, and there will be terms in the "tail" of the Taylor series that break this symmetry. This is a familiar feature of the Hopf bifurcation.

From equations (11) we obtain, to third order, the equations

\begin{align}
\dot{v} &= v[\lambda + i\omega + a|w|^2 + b(|v|^2 + |w|^2)] \\
\dot{w} &= w[\lambda - i\omega + \bar{a}|v|^2 + \bar{b}(|v|^2 + |w|^2)];
\end{align}

these equations may be written in terms of the amplitudes \(x_1, x_2\) and phases \(\phi_1, \phi_2\) defined by

\[ v = x_1 e^{i\phi_1}, \quad w = x_2 e^{i\phi_2}, \quad x_1, x_2 > 0, \]

as the four real equations

\begin{align}
\dot{x}_1 &= x_1[\lambda + a_x x_2^2 + b_x A^2] \\
\dot{x}_2 &= x_2[\lambda + a_x x_1^2 + b_x A^2] \\
\dot{\phi}_1 &= \omega + a_i x_2^2 + b_i A^2 \\
\dot{\phi}_2 &= -\omega - a_i x_1^2 - b_i A^2,
\end{align}
where subscripts \( r, i \) denote real and imaginary parts, respectively. We have introduced the total amplitude \( A \) given by

\[
A^2 = x_1^2 + x_2^2.
\]  

(17)

In the applications \( A^2 \) is to be identified with \( N - 1 \). Observe that the equations have decoupled into two amplitude equations and two phase equations. This is a general feature of this problem, and the resulting amplitude equations have \( D_4 \) symmetry [9].

The steady-state solutions of equations (16) are presented in Table 1 together with their stability assignments. There are no limit cycles.

<table>
<thead>
<tr>
<th>solution</th>
<th>equation</th>
<th>eigenvalues</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>( A^2 = 0 )</td>
<td>( \lambda, \lambda )</td>
<td>( \phi )</td>
</tr>
<tr>
<td>(x,0)</td>
<td>( \lambda + b_r A^2 = 0 )</td>
<td>( -2\lambda, -\lambda a_r/b_r )</td>
<td>TW</td>
</tr>
<tr>
<td>(x,x)</td>
<td>( \lambda + (\frac{1}{2} a_r + b_r) A^2 = 0 )</td>
<td>( -2\lambda, \lambda a_r/2 a_r + b_r )</td>
<td>SW</td>
</tr>
</tbody>
</table>

We see that there are two nonlinear solution branches, the TW and SW branches. We represent these as bifurcation diagrams \( A \) vs. \( \lambda \) in the \( a_r - b_r \) plane (Fig. 5). Observe that a stable solution branch exists only when both branches bifurcate supercritically, and that the stable branch is the one with the larger amplitude (cf. [10]).

These results are valid whenever the nondegeneracy conditions

\[
(18) \quad a_r \neq 0, \quad b_r \neq 0, \quad a_r + 2b_r \neq 0
\]

hold at \( \lambda = 0 \). However, the thermosolutal problem is degenerate. A simple calculation shows that \( b_r \equiv 0 \) [11, 9, 12], and within the system (2) it is not possible to make \( b_r \neq 0 \). This fact has important consequences for the stability of the solution branches. In order to analyze this situation we carry the above calculation to fifth order. From equations (11) we obtain the system

\[
(19a) \quad \dot{x}_1 = x_1 [\lambda + a_r x_2^2 + c_r A^4 + d_r x_2^2 A^2 + e_r x_2^4] + ...$

FIGURE 5. The Hopf bifurcation with $O(2)$ symmetry showing standing waves (SW) and travelling waves (TW) in the $a_r - b_r$ plane. The plane divides into six regions with qualitatively different bifurcation diagrams ($A$ vs. $\lambda$) in each. Continuous lines indicate stable solutions, dashed lines unstable ones.

$\dot{x}_2 = x_2 [\lambda + a_r x_1^2 + c_r A^4 + d_r x_1^2 A^2 + e_r x_1^4] + ...$

Again, there are two nontrivial steady-state branches whose properties are summarized in Table 2. We exhibit the results in the form of bifurcation diagrams in the $a_r - c_r$ plane (Fig. 6). Observe that now only the TW branch can be stable, and then only when both branches bifurcate supercritically, i.e., where $a_r < 0$, $c_r < 0$. The SW branch can never be stable near the bifurcation. This result is independent of $d_r$ and $e_r$, provided the nondegeneracy conditions

$$a_r \neq 0, \quad c_r \neq 0$$

hold at $\lambda = 0$. 
FIGURE 6. As in Fig. 5 but for the degenerate case $b_r \equiv 0$. Bifurcation diagrams are shown in the $a_r - c_r$ plane. The SW branch is never stable.

<table>
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<td>$\phi$</td>
</tr>
<tr>
<td>(x,0)</td>
<td>$\lambda + c_r A^4 = 0$</td>
<td>$-4\lambda, [a_r + (d_r + e_r)A^2]A^2$</td>
<td>TW</td>
</tr>
<tr>
<td>(x,x)</td>
<td>$\lambda + \frac{1}{2} a_r A^2 + (c_r + \frac{1}{2} d_r + \frac{1}{4} e_r) A^4 = 0$</td>
<td>$-4\lambda - a_r A^2, -[a_r + (d_r + e_r)A^2]A^2$</td>
<td>SW</td>
</tr>
</tbody>
</table>

Our numerical finding that standing waves are unstable to travelling waves suggests that for the parameter values employed in the numerical calculations ($\sigma = 1$, $\tau = 10^{-1/2}$, $R_S = 10^4$), $a_r$ and $c_r$ are both negative.

4. Discussion. We have described numerical calculations of travelling waves in two-dimensional thermosolutal convection with periodic boundary conditions in the horizontal,
and stress–free boundary conditions at top and bottom, and found numerically that standing waves are unstable to travelling waves. We have developed a theory using the symmetry properties introduced by the periodicity to explain the essential features of the numerical results. For the parameter values considered, the travelling wave solutions appear to be the only stable solutions near the Hopf bifurcation.

The theory offers a number of extensions and suggestions for numerical investigations that are currently under way. In particular, the theory suggests the study of parameter values for which the coefficient $a_r$ is small. The condition $a_r = 0$ is equivalent to

\begin{equation}
4\sigma \omega_0^4 - \omega_0^2 \tau \Delta (4 + 2\omega + \omega^2) + 2\omega \sigma (\Delta + \sigma \tau + \tau^2) - \omega^2 \Delta \tau (\Delta + \sigma \tau + \tau^2) = 0,
\end{equation}

where

\[ \Delta = 1 + \sigma + \tau, \quad \omega = 4\pi^2 / (\pi^2 + k^2) \]

[13]. It can be shown that by unfolding the doubly degenerate Hopf bifurcation with $O(2)$ symmetry, quasi–periodic waves with two and three independent frequencies can be produced in secondary and tertiary bifurcations, respectively [12, 14].

Of particular interest is the interaction of the TW and SW branches with the branch of steady states (SS) that bifurcates subcritically from the trivial solution at $R_T^c > R_T^c$. This problem can be studied by unfolding the Takens–Bogdanov bifurcation with $O(2)$ symmetry [16]. This bifurcation occurs at

\begin{equation}
R_T = \frac{\sigma + \tau}{\sigma(1 - \tau)} \frac{(\pi^2 + k^2)^3}{k^2}, \quad R_S = \frac{\tau^2 (\sigma + 1)}{\sigma(1 - \tau)} \frac{(\pi^2 + k^2)^3}{k^2}.
\end{equation}

The theory shows that the TW branch terminates on the SS branch at a steady–state bifurcation, with the travelling velocity $c$ approaching zero linearly. Thus the travelling wave slows down more and more as the steady state is approached. At a fixed point $z$ the period of oscillations in the vertical velocity $\Psi_z$ tends to infinity, but as $(R_W^{TW} - R_T)^{-1}$ in contrast to the SW branch which terminates in a heteroclinic orbit with the period approaching infinity as $-ln(R_W^{SW} - R_T)$ [16]. In addition, for parameters near (22) a secondary branch of quasi–periodic waves with two frequencies can bifurcate from the TW branch, and terminate either in a Hopf bifurcation on the SW branch, or in a saddle–loop bifurcation back on the TW branch. However, because of the degeneracy of the TW
branch, a higher order calculation must be carried out to establish these results for the present problem.

In a recent paper, Walden, Kolodner, Passner and Surko [17] have reported observations of travelling waves in an alcohol–water mixture heated from below. In such a fluid mixture the Soret effect sets up a concentration that behaves much like the externally imposed solute gradient discussed here. In fact, with boundary conditions of the form (3) the two problems can be transformed into one another [18]. In particular, in the binary fluid the coefficient \( b_r = 0 \) also [12], and hence we expect travelling waves to be the only stable solutions near the initial bifurcation. The travelling waves observed by Walden et al. resemble those described here, except of course that because of the finiteness of their system, rolls actually disappear at one end, with a new roll appearing at the opposite end of the container. This process introduces a modulation on the wave, with periodicity \( d/2c \). This effect is observed at the level of less than 1\%. However, in the experiments the travelling waves apparently bifurcate from finite amplitude steady convection, and hence are not described by the theory as presented here. On the other hand, the observation suggests that travelling waves that bifurcate from the conduction state should be observable in experiments in containers of sufficiently large aspect ratio.

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