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Amplitude equations for travelling wave convection

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Abstract. New asymptotically exact amplitude equations are derived for a dissipative system near a Hopf bifurcation. Unlike the usual coupled complex Ginzburg–Landau equations these are valid for \( \mathcal{O}(1) \) group velocities.

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The coupled complex Ginzburg–Landau equations form the basis for a number of studies of two-dimensional wave dynamics in doubly diffusive [1,2] and binary fluid convection [3,4]. Solutions of these equations correspond qualitatively with the results of several experiments [5–7], although detailed modelling of the nonlinear states has not been attempted. In particular the equations, when solved over a finite domain, exhibit the so-called ‘blinking’ [7] states in which the direction of propagation of a travelling wave reverses more or less periodically in time, and the ‘confined’ states [6] in which a travelling wave fills only part of the container, generally (though not always) near the sidewall towards which it travels.

In this paper we point out that the regime of validity of these equations is more limited than sometimes appreciated, and offers a consistent set of equations valid over a greater range of parameter values. Consider a translation and reflection invariant system with a translation and reflection invariant state that undergoes a symmetry-breaking Hopf bifurcation when a parameter \( R \) reaches a critical value \( R_c \). The linear stability properties of the trivial state are then described by the dispersion relation [8]

\[
F(s, k^2, R) = 0
\]

where \( s \) is the growth rate and \( k \) the horizontal wavenumber of the perturbation. The condition (1) yields two equations that determine the neutral curve \( R = R(k^2) \) and the corresponding Hopf frequency \( s = \pm i \Omega(k^2) \). Suppose that the neutral curve has a minimum at \( R = R_c, k = k_c \). Near \( R_c \) we may write

\[
R = R_c + \Delta R \quad s = \pm i \Omega + \Delta s \quad k = k_c + \Delta k
\]

and hence, at leading order,

\[
\Delta s F_s + \Delta k F_k + \Delta R F_R = 0.
\]

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Along the neutral curve $\Delta s$ is purely imaginary, $\Delta s = \pm i\Omega \Delta k$, while the condition that $R$ has a minimum at $R = R_c$ implies that $\Delta R/\Delta k = 0$. Thus

$$iF_i/F_s = \pm \Omega_k (R_c, k_c) \equiv \pm v_g$$

(4)

where the group velocity $v_g$ is real and typically $O(1)$. Now suppose that the stream function $\psi(x, z, t)$ has the asymptotic expansion

$$\psi = \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \ldots$$

(5)

where $\varepsilon$ is a small parameter such that $R = R_c + \mu \varepsilon^2$, $\mu = O(1)$, and

$$\psi_1 = \text{Re} \{ A_1(X, T) e^{i(k x + \Omega t)} + B_1(X, T) e^{i(k x - \Omega t)} \} f(z).$$

(6)

Here $f(z)$ is the vertical eigenfunction and $\Omega_z = \Omega(k_c)$. In the usual derivation of the coupled complex Ginzburg–Landau (CCGL) equations one assumes that the slow spatial and temporal scales are given by

$$X = \varepsilon x \quad T = \varepsilon^2 t$$

(7a)

and sets $k = k_c - i \varepsilon \partial_X$, $s = \pm i \Omega \varepsilon^2 \partial_T$. On substituting these expressions into the dispersion relation (1) and formally expanding in powers of $\varepsilon$ one finds, at $O(\varepsilon^2)$, an unbalanced term $v_g \partial_X A$, a difficulty that is typically resolved by assuming (implicitly) that $v_g = O(\varepsilon)$. If this assumption is made then at $O(\varepsilon^3)$ one obtains the usual (linearized) equations

$$\partial_T A_1 - (v_g/\varepsilon) \partial_X A_1 = \mu A_1 + \gamma A_{1XX}$$

$$\partial_T B_1 + (v_g/\varepsilon) \partial_X B_1 = \mu B_1 + \gamma B_{1XX}.$$  

(8a)

(8b)

As appropriate for an asymptotic expansion all the terms in equations (8) are independent of $\varepsilon$. The assumption that $v_g = O(\varepsilon)$ restricts, however, the applicability of (8) to the vicinity of the codimension-two point, where $\Omega_c = O(\varepsilon)$. This difficulty with the derivation of the CCGL equations has been remarked on by earlier authors [4], and arises in systems with two counterpropagating waves because the offending group velocity terms cannot both be removed by going to an appropriate comoving frame. Similar problems have been noted in non-dissipative systems [9].

It is possible to derive amplitude equations that are valid for $v_g = O(1)$, and hence away from the codimension-two point. To this end we introduce two slow times,

$$T = \varepsilon^2 t.$$  

(7b)

and assume that $A_1 = A_1(X, \tau, T)$, $B_1 = B_1(X, \tau, T)$. At $O(\varepsilon^2)$ we obtain

$$\partial_\tau A_1 - v_g \partial_X A_1 = 0, \quad \partial_\tau B_1 + v_g \partial_X B_1 = 0.$$  

(9)

It follows that $A_1 = A_1(\xi, T)$, $B_1 = B_1(\eta, T)$, where

$$\xi = v_g \tau + X, \quad \eta = v_g \tau - X.$$  

(10)
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To obtain evolution equations for $A_1, B_1$ we go to $\mathcal{O}(\varepsilon^3)$:

\begin{align}
\partial_T A_1 + (\partial_x \nu_A \partial_x) A_2 &= \mu A_1 + \gamma A_{1XX} + (a |A_1|^2 + b |B_1|^2) A_1 \\
\partial_T B_1 + (\partial_x \nu_B \partial_x) B_2 &= \mu B_1 + \gamma B_{1XX} + (\bar{a} |B_1|^2 + \bar{b} |A_1|^2) B_1
\end{align}

or, equivalently,

\begin{align}
2v_\xi \partial_{\xi} A_2 &= -\partial_T A_1 + \mu A_1 + \gamma A_{1\xi\xi} + (a |A_1|^2 + b |B_1|^2) A_1 \\
2v_\xi \partial_{\xi} B_2 &= -\partial_T B_1 + \mu B_1 + \gamma B_{1\eta\eta} + (\bar{a} |B_1|^2 + \bar{b} |A_1|^2) B_1
\end{align}

Expansion (5) remains asymptotic for times $t = \mathcal{O}(\varepsilon^{-2})$ only if an appropriate solvability condition holds. This condition, obtained by integrating (12a) over $\eta$ and (12b) over $\xi$, yields the desired evolution equations:

\begin{align}
\partial_T A_1 &= (\mu + b \lambda) A_1 + \gamma A_{1\xi\xi} + a |A_1|^2 A_1 \\
\partial_T B_1 &= (\mu + \bar{b} \nu) B_1 + \gamma B_{1\eta\eta} + \bar{a} |B_1|^2 B_1
\end{align}

where

\begin{align}
\lambda(T) &= (1/Q) \int_0^Q |B_1|^2 \, d\eta & \nu(T) &= (1/P) \int_0^P |A_1|^2 \, d\xi
\end{align}

and $P, Q$ are the periods of $A_1$ and $B_1$ in $\xi$ and $\eta$, respectively. Here $a = a_r + i a_i$, $b = b_r + i b_i$, and $\gamma = \gamma_r + i \gamma_i$ are complex coefficients. Note that equations (13) are formally uncoupled, although $\lambda$ and $\nu$ have to be determined self-consistently. Equations (13) and (14) are the correct asymptotic evolution equations when $v_\xi = \mathcal{O}(1)$ and replace the CCGL equations in this regime.

For spatially uniform patterns equations (13) and (14), like the CCGL equations, reduce to the normal form for the Hopf bifurcation with $O(2)$ symmetry [2,10]. The equations then admit three types of solution, the trivial one $(A_1, B_1) = (0,0)$, travelling waves $(Re^{i\omega T}, 0)$, $(0, Re^{-i\omega T})$ and standing waves $(Re^{i\omega T}, Re^{-i\omega T})$. The travelling waves (TW) are stable with respect to spatially uniform perturbations for $a_r < 0, a_r - b_r > 0$, while standing waves (SW) are stable for $a_r - b_r < 0, a_r + b_r < 0$. We can also examine the stability of these states with respect to small sideband perturbations. We then find, as for CCGL, that TW are modulationally unstable if

\begin{align}
a_r \gamma_r + a_i \gamma_i > 0 & \quad a_r < 0.
\end{align}

The instability sets in at long wavelengths first. For SW, however, equations (13) and (14) also yield the condition (15) for modulational instability, a result that differs from the CCGL condition [2].

It is worth noting that even with the simplest boundary conditions restricting the system geometry, $\psi = 0$ at $x = 0, \varepsilon$ [11], where $\varepsilon' = \mathcal{O}(1)$, equations (13) and (14) exhibit non-trivial solutions. These conditions imply that for all $\tau$

\begin{align}
A_1 (v_\xi \tau, T) + B_1^* (v_\xi \tau, T) &= 0 \\
A_1 (v_\xi \tau + \varepsilon \ell', T) e^{ik\ell'} + B_1^* (v_\xi \tau - \varepsilon \ell', T) e^{-ik\ell'} &= 0
\end{align}

The instability sets in at long wavelengths first. For SW, however, equations (13) and (14) also yield the condition (15) for modulational instability, a result that differs from the CCGL condition [2].
or, equivalently,
\[ A_1(v_g \tau + \epsilon \ell, T) = A_1(v_g \tau - \epsilon \ell, T) e^{-2i k \ell}. \]  
(16c)
The pattern dynamics depend on the degree to which the sidewalls compress or expand the natural wavelength of the rolls. This effect is measured by the ratio \( k_0 \ell / \pi = n + \delta_n \), where \( n \) is a (large) integer and \( 0 < \delta_n < 1 \). Condition (16c) implies that
\[ A_1(\xi, T) = e^{-ik_0 \xi} g_n(\xi, T) \]
\[ B_1(\eta, T) = -e^{ik_0 \eta} g_n^*(\eta, T) \]
where \( g_n \) is \( 2\epsilon \ell \)-periodic in \( \xi \) and \( k_n = \pi \delta_n / \epsilon \ell \). Note that \( A_1 \) and \( B_1 \) are quasiperiodic in \( \xi, \eta \). At leading order the stream function now takes the form
\[ \psi = \epsilon \text{Re} \left\{ \left[ g_n(v_g \tau + X, T) e^{i\Omega t} - g_n^*(v_g \tau - X, T) e^{-i\Omega t} \right] e^{ik' x} f(z) \right\} + \mathcal{O}(\epsilon^2) \]  
(18a)
where
\[ k' = k_0 - \epsilon k_n \quad \Omega' = \Omega_c - \epsilon v_g k_n. \]  
(18b)
In view of (17) \( \lambda = \nu \) and explicit solutions can readily be found in terms of elliptic functions. Suppose \( g_n(\xi) \) is peaked around \( \xi = 0 \). Then the stream function \( \psi \) describes a blinking state with period \( \tau_0 = 2\epsilon \ell / v_g \). To see this consider the average of \( |\psi|^2 \) over one convection cell and over the fast time \( t \):
\[ \langle |\psi|^2 \rangle = \left( 1/2 \right) (|g_n(v_g \tau + X, T)|^2 + |g_n(v_g \tau - X, T)|^2). \]  
(19)
At time \( \tau = 0 \) \( \langle |\psi|^2 \rangle \) has a peak at \( x = 0 \), i.e., at the left side of the box, while at \( \tau = \tau_0 / 2 \) it is peaked at \( x = \ell \), i.e., at the right side of the box. As a function of \( \tau \), \( \langle |\psi|^2 \rangle \) therefore describes periodic reversals in the envelope of the travelling pattern. When \( a, b, \gamma \) are real and \( a < 0, \gamma > 0 \), equations (13a, b) possess a Liapunov functional on the domain \( 0 \leq x \leq 2\ell \). Hence those solutions (18) for which \( g_n \) corresponds to a minimum of this functional are stable.

In figure 1 we show \( \langle |\psi|^2 \rangle \) as a function of \( \tau \) in the domain \( 0 \leq X \leq \epsilon \ell \) for the two cases (a) \( a + b < 0, a < 0 \) and (b) \( b < 0, a > 0 \). In both cases \( \gamma = 1 \) and the amplitude has been arbitrarily scaled to 1. In (a) \( |g_n(\xi)|^2 = \sin^2(\xi, k) \), with \( k \) determined by \( 2K(k) = \epsilon \ell / \pi \); in (b) \( |g_n(\xi)|^2 = \cos^2(\xi, k), k < k_n \). where \( k_n \) depends on \( a, b \).

The above calculation shows that blinking states are a natural consequence at small amplitude of imposing the boundary conditions \( \psi(0) = \psi(\ell) = 0 \) on a dispersive system with \( \mathcal{O}(1) \) group velocity. The prevalence of the blinking states is a consequence of the boundary conditions which do not allow partial reflection from the endwalls [4]. In particular the travelling waves which travel from one wall to the other for all time without reversals do not exist with these boundary conditions. In contrast the CCGL with more realistic boundary conditions admit small amplitude blinking states only if the coefficients \( a, b, \gamma \) are complex, and then only in a limited parameter regime [12]; at larger amplitudes blinking states occur even for real coefficients [4].

In this paper we have shown that when the group velocity \( v_g = \mathcal{O}(1) \) the usual CCGL equations governing the evolution of counterpropagating disturbances require important modification. In the formulation presented here we have obtained two equations for the wave amplitudes that are coupled only through their spatial averages. The resulting equations describe similar phenomena as the CCGL equations, including the blinking states, but in a larger parameter regime. A detailed discussion of their solutions with appropriate boundary conditions will be presented elsewhere.
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Figure 1. Plots of $|\psi|^2$ against $X$ at successive times $\tau$, expressed as fractions of the period $\tau_0$ of the blinking state. In (a) $|g_n(\xi)|^2 = \text{sn}^2(\xi, 0.99)$ and in (b) $|g_n(\xi)|^2 = \text{cn}^2(\xi, 0.99)$.

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