Oscillatory convection in a rotating layer

E. Knobloch and M. Silber

Department of Physics, University of California, Berkeley, CA 94720, USA
Department of Applied Mechanics, California Institute of Technology, Pasadena, CA 91125, USA

Received 18 June 1992
Accepted 11 August 1992
Communicated by R.S. MacKay

The stability of travelling and standing rolls in oscillatory rotating convection is considered from the point of view of equivariant bifurcation theory. To study stability with respect to oblique perturbations the problem is formulated on a rotating rhombic lattice. All primary solution branches with maximal isotropy are determined together with their stability properties using a truncation of the most general equivariant vector field at third order. In addition as many as seven branches of temporally periodic or quasiperiodic solutions with submaximal isotropy may be present. Instabilities analogous to the Küppers-Lortz instability of steady rolls in rotating convection are uncovered for both travelling rolls and for standing rolls. These instabilities are triggered by the formation of a heteroclinic orbit connecting two travelling roll states or two sets of standing rolls with different wave vectors. Conditions are given for the formation and asymptotic stability of a structurally stable heteroclinic cycle connecting four travelling roll states. The results are compared with recent studies of oscillatory patterns on a rotating square lattice and on a nonrotating rhombic lattice.

1. Introduction

Bifurcation theory has been of great value in understanding the selection process among spatially periodic patterns in a variety of continuous systems. The main reason for this success is its emphasis on the use of symmetries. The symmetries in question are those of the periodic lattice on which the patterns lie (either rhombic, square or hexagonal if the system is two-dimensional), together with any normal form symmetry. The trivial state of the system is, by definition, invariant under this group of symmetries, hereafter denoted by $\Gamma$. As a parameter is increased this state loses stability in a symmetry-breaking bifurcation forming a pattern, i.e., a state that is invariant under only a subgroup of $\Gamma$. The possible patterns are thus identified with the isotropy subgroups of the full symmetry group, thereby capturing precisely those properties that patterns in different systems have in common. In addition the requirement that the amplitude equations be equivariant with respect to the appropriate representation of the full symmetry group $\Gamma$ can be used to write down the general form of such equations for all systems with a common symmetry $\Gamma$. Moreover the residual symmetry of each pattern can be used with advantage to simplify the determination of its stability properties with respect to all the competing patterns on the lattice. These developments have recently been summarized by Golubitsky et al. [1] and their application to spatially periodic pattern selection in planar systems invariant under the Euclidean group $E(2)$ of rotations, reflections and translations of a plane has been reviewed by Crawford and Knobloch [2].

When reflection symmetry is broken so that the system is invariant under translations and proper rotations only, a new phenomenon may arise. This is an instability, discovered by Küppers and Lortz [3] in their study of con-
vection in a rotating fluid layer, that causes a steady roll pattern to be unstable with respect to an identical pattern rotated through some angle $\Theta$ (in the corotating frame of reference). Such an instability is absent in a nonrotating system and requires the rotation rate to exceed a critical value. Note that for low rotation frequencies the centrifugal force may be neglected so that rotation manifests itself primarily through the Coriolis force \[^{[4]}\]. Consequently, the system remains invariant under translations and proper rotations, but not reflections; the symmetry of the problem is then that of the special Euclidean group $\text{SE}(2)$.

To understand the origin of the Kuppers–Lortz instability consider a steady state bifurcation on a rhombic lattice defined by the fundamental lattice vectors $\mathbf{a}_1, \mathbf{a}_2$ such that $\mathbf{a}_1 \cdot \mathbf{a}_2 = |\mathbf{a}_1||\mathbf{a}_2| \cos \Theta$ with $|\mathbf{a}_1| = |\mathbf{a}_2| = 2\pi/k_c$. Here $k_c$ is the critical wavenumber of the instability. Any smooth field $\psi(x,y,t)$ describing the state of the fluid and satisfying the periodicity requirement

$$\psi(x + n_1\mathbf{a}_1 + n_2\mathbf{a}_2, y, t) = \psi(x, y, t),$$

$$(n_1, n_2) \in \mathbb{Z}^2,$$  \hspace{1cm} (1)

can be written in the form

$$\psi = \sum_{m,n=0}^{\infty} z_{mn}(t) \exp(i(mk_1 + nk_2)x) \, Y_{mn}(y)$$
$$+ \text{c.c.}$$  \hspace{1cm} (2)

Here $k_i \cdot a_j = 2\pi\delta_{ij}, i, j = 1, 2$, and $Y_{mn}(y)$ denotes the vertical structure of the $(m,n)$-mode. Near onset we write

$$\psi = (z_1 \exp(ik_1x) + z_2 \exp(ik_2x))Y(y) + \ldots$$
$$+ \text{c.c.}$$  \hspace{1cm} (3)

with the amplitudes of the higher order modes slaved to the evolution of the amplitudes $z_1, z_2$. In a system invariant under proper rotations and translations, the symmetry group associated with patterns doubly-periodic on a rhombic lattice is $\mathbb{Z}_2 \times \mathbb{T}^2$, where $\mathbb{Z}_2$ corresponds to a rotation by $\pi$ and $\mathbb{T}^2$ is the two-torus of translations. The symmetry group $\mathbb{Z}_2 \times \mathbb{T}^2$ acts on $(z_1, z_2) \in \mathbb{C}^2$ as follows:

rotation $x \rightarrow -x$:

$$(z_1, z_2) \rightarrow (z_1, z_2),$$

translations $x \rightarrow x + d$:

$$(z_1, z_2) \rightarrow (z_1 \exp(ik_1d), z_2 \exp(ik_2d)).$$  \hspace{1cm} (4)

The amplitude equations are equivariant with respect to these operations and hence take the form

$$\dot{z}_1 = f_1(\lambda, |z_1|^2, |z_2|^2)z_1,$$
$$\dot{z}_2 = f_2(\lambda, |z_1|^2, |z_2|^2)z_2,$$  \hspace{1cm} (5)

where $\lambda$ is the bifurcation parameter, and $f_j, j = 1, 2$, are real functions that satisfy the additional constraint $f_1(\lambda, u, 0) = f_2(\lambda, 0, u)$. This constraint follows from the $\text{SO}(2) \subset \text{SE}(2)$ invariance of the original physical problem: the evolution equations for pure rolls in the $k_1$-direction ($z_2 = 0$) must agree with those obtained for pure rolls in the $k_2$-direction ($z_1 = 0$). In terms of the real variables $z_j = r_j \exp(i\phi_j), j = 1, 2$, we therefore have

$$\dot{r}_1 = \lambda r_1 + ar_1^3 + br_1^2r_1 + \ldots, \quad \dot{\phi}_1 = 0,$$
$$\dot{r}_2 = \lambda r_2 + cr_2^2r_2 + ar_2^3 + \ldots, \quad \dot{\phi}_2 = 0,$$  \hspace{1cm} (6)

near $\lambda = 0$. Pure rolls correspond to solutions with $(r_1, r_2) = (r, 0)$, or $(r_1, r_2) = (0, r)$, $r^2 = -\lambda/a$ (hereafter $R_1, R_2$, respectively). The stability properties of these two sets of rolls are not identical. This is because the presence of the Coriolis force breaks the reflection symmetry $(z_1, z_2) \rightarrow (z_2, z_1)$ that forces the coefficients $b, c$ to be identical in a nonrotating system.
Fig. 1. Typical phase portraits showing the formation of the heteroclinic orbit responsible for the onset of the Küppers–Lortz instability of steady rotating convection. (a) The nonrotating case ($b - c < a < 0$), with the stable and unstable manifolds of the mixed mode indicated in the $(r_1, r_2)$-plane. (b) The rotating case, no Küppers–Lortz instability ($b < c < a < 0$), and (c) the rotating case with a Küppers–Lortz instability from $R_1$ to $R_2$ ($b < a < c$, $a < 0$). The mixed mode no longer exists.

$a < 0$ so that both sets of rolls bifurcate super-critically, the stability of rolls $R_1$ with respect to perturbations in the form of $R_2$ is determined by $\text{sgn}(c - a)$, while the sign of the corresponding eigenvalue of $R_2$ is given by $\text{sgn}(b - a)$. For slow rotation ($b \approx c$) both eigenvalues are negative, but for large enough rotation $c - a$ becomes positive with the result that $R_1$ are now unstable to $R_2$, and by the same argument $R_2$ are unstable to a third set of rolls again at an angle $\Theta$, etc. The rotation rate at which this instability first sets in for some angle $\Theta$ was calculated by Küppers and Lortz for rotating Rayleigh–Bénard convection with stress-free boundaries [3], and by Küppers for no-slip boundaries [5]. Mathematically, the instability sets in when an unstable mixed mode $(r_1, r_2)$, $r_1 \neq r_2$, $r_1 r_2 \neq 0$, annihilates with $R_1$ in a subcritical pitchfork bifurcation, forming a heteroclinic orbit connecting $R_1$ to $R_2$ (see fig. 1). This first occurs for a particular angle $\Theta$ which we choose to define the lattice. For more details, see Knobloch and Silber [6]. The resulting instability has been observed experimentally [7,8].

The present paper is motivated by the above scenario. We seek analogous instabilities in a rotating system undergoing a Hopf bifurcation. Our analysis is pertinent to rotating Rayleigh–Bénard convection in low Prandtl number fluids; a linear analysis shows that in such a fluid the spatially uniform heat conduction state may lose stability via Hopf bifurcation if the rotation rate is sufficiently great [4]. In an earlier paper [9] we describe the analogue of the Küppers–Lortz instability for travelling rolls on a rotating square lattice ($\Theta = \pi/2$). The instability takes the form of a structurally stable attracting heteroclinic cycle connecting waves travelling in four orthogonal directions. A similar instability for standing rolls is, however, prohibited for the square lattice. In this paper we generalize the discussion to a Hopf bifurcation on a rotating rhombic lattice. We find that there are as many as eleven primary solution branches that bifurcate at the Hopf bifurcation. Four of these are always present and are periodic in time. Existence of the remaining seven depends on the values of the cubic coefficients; four of these solutions correspond to temporally-periodic standing wave patterns, two correspond to quasiperiodic travelling patterns, while the remaining solution corresponds to a quasiperiodic standing wave. We compute the stability properties of the four solutions that always exist, as well as those of the two quasiperiodic travelling patterns. We focus on Küppers–Lortz type instabilities for both travelling and standing rolls, and identify a structurally stable heteroclinic cycle between travelling rolls.

The paper is organized as follows. In section 2 we present the group-theoretic analysis of the problem, constructing the lattice of isotropy subgroups that describes the possible symmetries of patterns on the rotating rhombic lattice. In section 3 we construct the amplitude equations, truncated at third order, and compute explicitly
the primary branches. In section 4 we investigate their stability properties. In section 5 we determine the possible heteroclinic cycles in our system and investigate the possibility of a primary bifurcation to a quasiperiodic standing wave solution. In the last section we compare our results to those for the rotating square lattice [9] and the nonrotating rhombic lattice [10]. We also discuss the relationship between the rotating rhombic lattice and the rotating hexagonal lattice [11]. Finally, we discuss the application of our analysis to spatially periodic pattern formation in rotating convection.

2. Group-theoretic considerations

The symmetry group for patterns doubly-periodic on a (rotating) rhombic lattice is $Z_2 \times T^2$, where $Z_2$ is the reflection group generated by rotations of the lattice by $\pi$ and $T^2$ is the two-torus of translations (cf. eq. (4)). When the initial instability is a Hopf bifurcation near-identity nonlinear coordinate changes can be performed to put the resulting amplitude equations into Poincaré-Birkhoff normal form. This normal form has an additional $S^1$ temporal symmetry, which we refer to as the phase shift symmetry. In the following we therefore assume that the symmetry group of the problem is $\Gamma \equiv Z_2 \times T^2 \times S^1$. At onset we write (cf. eq. (3))

$$\psi(x, y, t) = (z_1(t) e^{ik_1 \cdot x} + z_2(t) e^{ik_2 \cdot x} + z_3(t) e^{-ik_1 \cdot x} + z_4(t) e^{-ik_2 \cdot x}) Y(y) + \cdots + \text{c.c.,}$$

and assume that the linear stability problem takes the form $\dot{z} = \mu(\lambda) z$, where $z \equiv (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, and $\mu(0) = i\omega_0$, $\text{Re}(\mu'(0)) > 0$. Here $\lambda$ is the bifurcation parameter. It follows that the quantities $(|z_1|, |z_3|)$ and $(|z_2|, |z_4|)$ represent amplitudes of left- and right-travelling waves in the $k_1$, $k_2$ directions, respectively. To understand how the group $\Gamma$ acts on $z \in \mathbb{C}^4$, we consider the effect of translations $x \rightarrow x + d$, and of rotations $x \rightarrow -x$ on the critical Fourier modes. We also assume an overall phase shift symmetry associated with the normal form for Hopf bifurcation [12]; this symmetry is interpreted as a time translation symmetry of periodic solutions with frequency $\delta \omega$: $t \rightarrow t + \phi/\delta \omega$. From (7) it follows that these operations are equivalent to the following action of $\Gamma$ on $\mathbb{C}^4$:

$$\kappa: \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \rightarrow \begin{pmatrix} z_3 \\ z_4 \\ z_1 \\ z_2 \end{pmatrix}, \quad \kappa \in \mathbb{Z}_2,$$

$$(\theta_1, \theta_2): \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\theta_1} z_1 \\ e^{i\theta_2} z_2 \\ e^{-i\theta_1} z_3 \\ e^{-i\theta_2} z_4 \end{pmatrix}, \quad (\theta_1, \theta_2) \in T^2,$$

$$\phi: z \rightarrow e^{i\phi} z, \quad \phi \in S^1.$$
E. Knobloch, M. Silber / Oscillatory convection in a rotating layer 217

Borhood of the Hopf bifurcation, must be \( \Gamma \)-equivariant:

\[
yf (z) = f (yz), \quad \text{for all } y \in \Gamma.
\] (9)

Equivariance with respect to \( T^2 \times S^1 \) determines the form of \( \dot{z}_1 \) and \( \dot{z}_2 \) [14]. Equivariance with respect to \( Z_2 \) then generates \( \dot{z}_3 \), \( \dot{z}_4 \) from \( \dot{z}_1 \), \( \dot{z}_2 \), respectively. It is thus straightforward to show that a smooth vector field commuting with the above symmetries has the form

\[
\begin{align*}
\dot{z}_1 &= p(|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2, Q) z_1 + q(|z_1|^2, |z_1|^2, |z_3|^2, |z_4|^2, \overline{Q}) z_2 z_4 z_3 z_2 \\
\dot{z}_2 &= r(|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2, \overline{Q}) z_2 \\
\dot{z}_3 &= s(|z_1|^2, |z_1|^2, |z_3|^2, |z_4|^2, Q) z_3 z_4 z_2 \\
\dot{z}_4 &= r(|z_3|^2, |z_4|^2, |z_1|^2, |z_2|^2, \overline{Q}) z_4 + s(|z_3|^2, |z_4|^2, |z_3|^2, |z_2|^2, Q) z_1 z_3 z_2,
\end{align*}
\] (10)

where \( Q = z_1 \overline{z}_2 \overline{z}_3 \overline{z}_4 \), and \( p, q, r \) and \( s \) are complex-valued functions of their arguments and of the bifurcation parameter \( \lambda \). Moreover, because of the \( \text{SO}(2) \) invariance of the original problem, the functions \( p \) and \( r \) are related by the constraint \( p(u, 0, v, 0, 0) = r(0, u, 0, v, 0) \).

We now summarize the information about the possible periodic solutions to (10) that can be obtained using group-theoretic arguments alone.

Each nontrivial (pattern-forming) solution \( z \neq 0 \) breaks the full symmetry \( \Gamma \), and therefore has symmetry characterized by an isotropy subgroup \( \Sigma_z \) of \( \Gamma \):

\[
\Sigma_z = \{ \sigma \in \Gamma: \sigma z = z \}. \quad (11)
\]

The successive breaking of the full symmetry is summarized by the lattice of isotropy subgroups of \( \Gamma \) shown in fig. 2. Associated with each isotropy subgroup is a linear subspace \( \text{Fix}(\Sigma) \), which is invariant under the dynamics (10): \( \text{Fix}(\Sigma) = \{ z \in C^4 : \sigma z = z, \forall \sigma \in \Sigma \} \). (12)

Hence all solutions with symmetry \( \Sigma \subset \Gamma \) evolve under (10) restricted to the subspace \( \text{Fix}(\Sigma) \). In table 1 we list all the isotropy subgroups of \( \Gamma \) (up to conjugacy) together with their fixed point subspaces. We may identify solutions in \( \text{Fix}(\Sigma) \) with spatio-temporal patterns using the expression (7). Note, however, that (7) represents the linear solution, which provides a good description of the pattern close to onset only, while the classification of the patterns in terms of the isotropy subgroups is valid even in the presence of nonlinearities. (See [15] for a discussion of linear steady rotating convection patterns that have more symmetry than expected from a direct calculation of the appropriate isotropy subgroup.)

The equivariant Hopf theorem guarantees the existence of primary solution branches with twodimensional fixed point subspaces [12]. In the present case these are the travelling rolls (\( \text{TR}_1 \), \( \text{TR}_2 \)), and the standing rolls (\( \text{SR}_1 \), \( \text{SR}_2 \)). We note that for the rotating square lattice problem (with \( Z_4 \times T^2 \times S^1 \) symmetry) the equivariant Hopf theorem determines the existence of two additional primary branches, referred to as standing squares (SS) and alternating rolls (AR) [9]. In the case of the nonrotating rhombic lattice (with \( D_2 \times T^2 \times S^1 \) symmetry), the equivariant Hopf theorem guarantees two travelling rectangle patterns in addition to TR, SR and AR; the SS, which become standing rectangles, must also exist [10]. All of these solutions are periodic in time. Note also that the rotating rhombic lattice problem approaches the rotating square lattice problem as \( \theta \rightarrow \pi/2 \). Similarly, as the rotation rate goes to zero, the discrete symmetry of the problem increases from \( Z_2 \) to \( D_2 \), the symmetry group of a rectangle. All of these problems may thus be viewed as perturbations of the Hopf bifurcation problem with
Fig. 2. Lattice of isotropy subgroups. Arrows indicate inclusion. Isotropy subgroups are described in tables 1 and 2.

Table 1
Fixed point subspaces corresponding to the different isotropy subgroups of $\Gamma$. The coordinates specifying each $\text{Fix}(\Sigma)$ are $(z_1, z_2, z_3, z_4)$ as in (7), each $v_j \in \mathbb{C}$, $j = 1, 2, 3, 4$. Table 2 describes each isotropy subgroup $\Sigma$. The trivial symmetry $Z_2^c$ is contained in every $\Sigma$.

<table>
<thead>
<tr>
<th>Name</th>
<th>$\text{Fix}(\Sigma)$</th>
<th>$\Sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>trivial solution $(T)$</td>
<td>$z = (0,0,0,0)$</td>
</tr>
<tr>
<td>I</td>
<td>travelling rolls $(TR_1)$</td>
<td>$z = (v_1,0,0,0)$</td>
</tr>
<tr>
<td>II</td>
<td>travelling rolls $(TR_2)$</td>
<td>$z = (0,v_1,0,0)$</td>
</tr>
<tr>
<td>III</td>
<td>standing rolls $(SR_1)$</td>
<td>$z = (v_1,v_1,0,0)$</td>
</tr>
<tr>
<td>IV</td>
<td>standing rolls $(SR_2)$</td>
<td>$z = (0,v_1,v_1,v_1)$</td>
</tr>
<tr>
<td>V</td>
<td>modulated rolls $(MR_1)$</td>
<td>$z = (v_1,0,v_2,0)$</td>
</tr>
<tr>
<td>VI</td>
<td>modulated rolls $(MR_2)$</td>
<td>$z = (0,v_1,0,v_7)$</td>
</tr>
<tr>
<td>VII</td>
<td>travelling bimodal $(TB_1)$</td>
<td>$z = (v_1,v_2,0,0)$</td>
</tr>
<tr>
<td>VIII</td>
<td>travelling bimodal $(TB_2)$</td>
<td>$z = (v_1,0,v_2,v_2)$</td>
</tr>
<tr>
<td>IX</td>
<td>standing bimodal $(SB)$</td>
<td>$z = (v_1,v_2,v_1,v_2)$</td>
</tr>
<tr>
<td>X</td>
<td>triply periodic $(QP3)$</td>
<td>$z = (v_1,v_2,v_3,v_4)$</td>
</tr>
</tbody>
</table>
Table 2
Generators of isotropy subgroups in table 1. Group elements of \( T^* \times S^1 \) are denoted \([ (\theta_1, \theta_2), \phi] \), where \((\theta_1, \theta_2) \in T^2 \) and \( \phi \in S^1 \).

<table>
<thead>
<tr>
<th>( \Sigma )</th>
<th>Generators of ( \Sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( TR_1 )</td>
<td>( SO(2) \times SO(2) )</td>
</tr>
<tr>
<td>( TR_2 )</td>
<td>( SO(2) \times SO(2) )</td>
</tr>
<tr>
<td>( SR_1 )</td>
<td>( Z_2 \times \mathbb{Z}_2 \times SO(2) )</td>
</tr>
<tr>
<td>( SR_2 )</td>
<td>( Z_2 \times \mathbb{Z}_2 \times SO(2) )</td>
</tr>
<tr>
<td>( MR_1 )</td>
<td>( Z_2 \times SO(2) )</td>
</tr>
<tr>
<td>( MR_2 )</td>
<td>( Z_2 \times SO(2) )</td>
</tr>
<tr>
<td>( TB_1 )</td>
<td>( SO(2) )</td>
</tr>
<tr>
<td>( TB_2 )</td>
<td>( SO(2) )</td>
</tr>
<tr>
<td>( SB )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( QP3 )</td>
<td>( \mathbb{Z}_2^c )</td>
</tr>
</tbody>
</table>

\( D_4 \times T^2 \times S^1 \) symmetry analyzed in detail by Silber and Knobloch [14]. Moreover, most of the submaximal isotropy solutions that are found in the next section may be understood as continuations of maximal isotropy solutions of the nonrotating rhombic lattice problem analyzed by Silber et al. [10].

3. The amplitude equations and primary branches

In the following we expand the functions \( p, q, r \) and \( s \) in (10) in a Taylor series about the origin, and truncate the resulting equations at cubic order:

\[
\begin{align*}
\dot{z}_1 &= \mu z_1 + (a|z_1|^2 + b|z_3|^2 + c|z_2|^2 + d|z_4|^2)z_1 \\
&+ ez_2z_4z_3, \\
\dot{z}_2 &= \mu z_2 + (a|z_2|^2 + b|z_4|^2 + f|z_1|^2 + g|z_3|^2)z_2 \\
&+ h z_1z_3z_4, \\
\dot{z}_3 &= \mu z_3 + (a|z_3|^2 + b|z_1|^2 + c|z_4|^2 + d|z_2|^2)z_3 \\
&+ ez_2z_4z_1, \\
\dot{z}_4 &= \mu z_4 + (a|z_4|^2 + b|z_2|^2 + f|z_3|^2 + g|z_1|^2)z_4 \\
&+ h z_1z_3z_2, \\
\end{align*}
\]

The coefficients \( \mu, a, b, c, d, e, f, g \) and \( h \) are all complex. In the following we suppose without loss of generality that \( \mu(\lambda) = \lambda + i\omega_c \), where \( \omega - \omega_c = O(\lambda) \) and \( \omega_c \) is the Hopf frequency at \( \lambda = 0 \). We note that for the rotating square lattice there is an additional \( \pi/2 \) rotational symmetry that forces \((c, d, e) = (g, f, h) \) [9], whereas for the nonrotating rhombic lattice reflection symmetry guarantees that \((c, d, e) = (f, g, h) \) [10]. In the case of a nonrotating square lattice \( c = d = f = g \) and \( e = h \) [14].

The small amplitude temporally-periodic solutions created at the Hopf bifurcation satisfy the equation \( \dot{z} = i\Omega z \), where the frequency \( \Omega \approx \omega_c \) is to be determined. Thus they satisfy the algebraic equations

\[
\begin{align*}
\nu z_1 + (a|z_1|^2 + b|z_3|^2 + c|z_2|^2 + d|z_4|^2)z_1 \\
&+ ez_2z_4z_3 = 0, \\
\nu z_2 + (a|z_2|^2 + b|z_4|^2 + f|z_1|^2 + g|z_3|^2)z_2 \\
&+ h z_1z_3z_4 = 0, \\
\nu z_3 + (a|z_3|^2 + b|z_1|^2 + c|z_4|^2 + d|z_2|^2)z_3 \\
&+ ez_2z_4z_1 = 0, \\
\nu z_4 + (a|z_4|^2 + b|z_2|^2 + f|z_3|^2 + g|z_1|^2)z_4 \\
&+ h z_1z_3z_2 = 0.
\end{align*}
\]
where \( \nu = \mu(\lambda) - i\Omega \). There are four solutions that reside in the two-dimensional fixed point subspaces. These take the form of travelling and standing rolls in the \( k_j \)-directions \((j = 1, 2)\), hereafter denoted by \( \text{TR}_j \) and \( \text{SR}_j \):

\[
\begin{align*}
\text{TR}_1 : \quad & (0, 0, 0, 0), \quad \nu + a|v|^2 = 0, \\
\text{TR}_2 : \quad & (0, 0, 0, 0), \quad \nu + a|v|^2 = 0, \\
\text{SR}_1 : \quad & (0, v, 0, 0), \quad \nu + (a + b)|v|^2 = 0, \\
\text{SR}_2 : \quad & (0, v, 0, 0), \quad \nu + (a + b)|v|^2 = 0.
\end{align*}
\]

These solutions have maximal isotropy (cf. fig. 2) and their existence is guaranteed by the equivariant Hopf theorem \([12]\). The branching equations are identical for each of the \( \text{TR}_j \) and each of the \( \text{SR}_j \) because of the underlying \( SO(2) \) symmetry of the unrestricted problem. However these pairs are not related by a symmetry associated with the rhombic lattice.

We next examine the dynamics in each of the four-dimensional fixed point subspaces listed in table 1 (entries V–IX). The dynamics in the modulated rolls subspaces \( \text{MR}_1, \text{MR}_2 \) are the same as in the \( O(2) \) Hopf bifurcation problem. This is readily verified by setting \( z = (v_1, 0, v_2, 0) \) and \( z = (0, v_1, 0, v_2) \) in (13) to obtain

\[
\begin{align*}
\dot{v}_1 &= \mu v_1 + (a|v_1|^2 + b|v_2|^2)v_1, \\
\dot{v}_2 &= \mu v_2 + (a|v_2|^2 + b|v_1|^2)v_2.
\end{align*}
\]

Generically, the only nontrivial periodic or quasiperiodic solutions in this subspace are \( \text{TR} \) and \( \text{SR} \) \([12]\). Thus there are no new solution branches associated with the \( \text{MR}_1 \) and \( \text{MR}_2 \) subspaces.

-associated with the four-dimensional fixed point subspace \( z = (v_1, v_2, 0, 0) \) is a quasiperiodic solution called a travelling bimodal pattern (TB1). The TB1 take the form \( (z_1, z_2, z_3, z_4) = (e^{i\Omega_1 t} \tilde{v}_1, e^{i\Omega_2 t} \tilde{v}_2, 0, 0) \). The amplitudes \( |\tilde{v}_1| \) and frequencies \( \Omega_j, j = 1, 2 \), are determined by

\[
\begin{align*}
\nu_1 + a|\tilde{v}_1|^2 + c|\tilde{v}_2|^2 &= 0, \\
\nu_2 + a|\tilde{v}_2|^2 + f|\tilde{v}_1|^2 &= 0,
\end{align*}
\]

where \( \nu_j = \mu - i\Omega_j \). Written in terms of their real and imaginary parts eqs. (17) become

\[
\begin{align*}
\lambda + a_1|\tilde{v}_1|^2 + c_1|\tilde{v}_2|^2 &= 0, \\
\lambda + a_2|\tilde{v}_2|^2 + f_1|\tilde{v}_1|^2 &= 0, \\
\omega - \Omega_1 + a_1|\tilde{v}_1|^2 + c_1|\tilde{v}_2|^2 &= 0, \\
\omega - \Omega_2 + a_2|\tilde{v}_2|^2 + f_1|\tilde{v}_1|^2 &= 0,
\end{align*}
\]

where the \( r \) and \( i \) subscripts specify the real and imaginary parts of the coefficients. The first two equations are identical to those for the steady state patterns on the rotating rhombic lattice (cf. equation 6). It follows that TB1 are given by

\[
\left( \begin{array}{c}
|\tilde{v}_1|^2 \\
|\tilde{v}_2|^2 
\end{array} \right) = \frac{-\lambda}{a_1^2 - c_1 f_1} \left( \begin{array}{c}
a_1 - c_1 \\
a_2 - f_1 
\end{array} \right); 
\]

the last two equations of (18) determine the corresponding frequencies \( (\Omega_1, \Omega_2) \). Note that these solutions exist if and only if \( a_1 - c_1 \) and \( a_2 - f_1 \) have the same sign, and hence are not present for all coefficient values.

The TB2 take the form \( (z_1, z_2, z_3, z_4) = (e^{i\Omega_1 t} \tilde{v}_1, 0, 0, e^{i\Omega_2 t} \tilde{v}_2) \), where

\[
\begin{align*}
\nu_1 + a|\tilde{v}_1|^2 + d|\tilde{v}_2|^2 &= 0, \\
\nu_2 + a|\tilde{v}_2|^2 + g|\tilde{v}_1|^2 &= 0,
\end{align*}
\]

The branching equations and the corresponding frequencies can both be deduced from those for TB1 on letting \( (c, f) \to (d, g) \). Thus TB2 exist if and only if \( (a_1 - d_1) (a_2 - g_1) > 0 \).

Next we consider the existence of more complicated but periodic solutions in the standing bimodal subspace (SB). Such solutions are obtained by setting \( z = (v_1, v_2, 0, 0) \) in (13) yielding the equations

\[
\begin{align*}
\nu_1 + a|\tilde{v}_1|^2 + d|\tilde{v}_2|^2 &= 0, \\
\nu_2 + a|\tilde{v}_2|^2 + g|\tilde{v}_1|^2 &= 0.
\end{align*}
\]
\[ \dot{v}_1 = \mu v_1 + (a + b)(|v_1|^2 + |v_2|^2)v_1 - m|v_2|^2 v_1 + e\overline{v}_1 v_2, \]
\[ \dot{v}_2 = \mu v_2 + (a + b)(|v_1|^2 + |v_2|^2)v_2 - n|v_1|^2 v_2 + hv_1^2 \overline{v}_2, \]
where
\[ m \equiv a + b - c - d, \quad n \equiv a + b - f - g. \]

Following Swift, we introduce the coordinate transformation [16]
\[ v_1 = \sqrt{r} \cos(\theta/2) e^{i(\varphi + \psi)/2}, \]
\[ v_2 = \sqrt{r} \sin(\theta/2) e^{-i(\varphi - \psi)/2}, \]
where \( \theta \in [0, \pi] \), and \( \varphi, \psi \in [0, 2\pi) \). Due to the \( S^1 \) symmetry of (21), the \( \psi \) equation decouples from the following equations for \( r, \theta, \varphi \):
\[ \dot{r} = 2r(\lambda + rg_0(\theta, \varphi)), \]
\[ \dot{\theta} = r \sin \theta \cdot g_1(\theta, \varphi), \]
\[ \dot{\varphi} = rg_2(\theta, \varphi), \]
where
\[ g_0 = a_r + b_t - \frac{1}{4} \sin^2 \theta \times \text{Re}(m + n - e^{-2i\varphi} - h e^{2i\varphi}), \]
\[ g_1 = \sin^2(\theta/2) \text{Re}(m - e^{-2i\varphi}) - \cos^2(\theta/2) \text{Re}(n - h e^{2i\varphi}), \]
\[ g_2 = -\sin^2(\theta/2) \text{Im}(m - e^{-2i\varphi}) + \cos^2(\theta/2) \text{Im}(n - h e^{2i\varphi}). \]

Note that the vector field (24)-(25) is invariant under \( \varphi \rightarrow \varphi + \pi \). This symmetry is a manifestation of the translation symmetry \( (v_1, v_2) \rightarrow (v_1, -v_2) \) of (21) in the spherical coordinates (23). By introducing the new time \( \tau \) that satisfies \( \dot{r} = r, \tau(0) = 0 \), we obtain the following dynamical system on the sphere \( S^2 \):
\[ \theta' = \sin \theta \cdot g_1(\theta, \varphi), \quad \varphi' = g_2(\theta, \varphi), \]
where \( \cdot \) denotes differentiation with respect to \( \tau \). A fixed point \( (r_0, \theta_0, \varphi_0) \) of (24) corresponds to a periodic solution of (13) since \( \psi = 2\omega + \cdots \). Note that the bifurcation parameter \( \lambda \) does not appear in (26). Whether a periodic solution in the SB subspace bifurcates supercritically or subcritically is determined by the sign of \( g_0(\theta_0, \varphi_0) \) in the \( r \) equation since
\[ r_0 = -\lambda/g_0(\theta_0, \varphi_0) > 0. \]

The periodic solutions \( SR_1, SR_2 \) appear as fixed points of (26) at the north and south poles \( (\theta = 0, \pi) \). All other periodic solutions in the SB subspace correspond to solutions of \( g_1(\theta, \varphi) = g_2(\theta, \varphi) = 0 \). From these conditions we obtain the following two equations for \( \tan^2(\theta/2) \):
\[ \tan^2(\theta/2) = \frac{\text{Re}(n - h e^{2i\varphi})}{\text{Re}(m - e^{-2i\varphi})} > 0 \\
+ \frac{\text{Im}(n - h e^{2i\varphi})}{\text{Im}(m - e^{-2i\varphi})} > 0. \]

Thus for submaximal periodic solutions to exist in the SB subspace, there must exist a \( \varphi \) satisfying the equation
\[ \text{Im}[\{n - h e^{2i\varphi}\} (\overline{m} - \overline{e} e^{2i\varphi})] = 0, \]
subject to the requirement \( \tan^2(\theta/2) > 0 \). The existence conditions (28), (29) have the following geometric interpretation if we view \( n - h e^{2i\varphi} \) and \( m - e^{-2i\varphi} \) as vectors in the complex plane. Equation (29) specifies that for some angle \( \varphi \) the vectors must be parallel (or antiparallel) for a solution to exist. The inequalities then rule out the case where the vectors are antiparallel. Given this interpretation, it is straightforward to find regions of the \( (m, n, e, h) \)-coefficient space where no submaximal periodic solutions exist (see, for example, fig. 3). Some of the interesting dynamics associated with these regions are examined in section 5.
Fig. 3. Possible points \( n - h \, e^{2i\varphi} \) and \( m - e \, e^{-2i\varphi} \) are plotted in the complex plane for all values of \( \varphi \in [0, \pi] \). Note that vectors from the origin to points on the two circles are never parallel. In (a) \( g_1(\theta, \varphi) > 0 \) in (26) for all values of \( \theta, \varphi \), and in (b) \( g_2(\theta, \varphi) < 0 \). In both cases, no submaximal periodic solutions exist in the SB subspace.

A complete classification of the solutions of (29) in the \((m, n, e, h)\)-coefficient space is beyond the scope of the present paper. However, in the special case where \( m = n, e = h \) the dynamics of the associated spherical system (26) have been determined [16]. In this case there are either two or four distinct solutions, each of which comes in a symmetric pair because of the symmetry \( \varphi \rightarrow \varphi + \pi \) of (25): one of the pair is in the hemisphere \( \varphi \in [0, \pi) \) and the other satisfies \( \varphi \in [\pi, 2\pi) \). For these special coefficient values the solutions satisfy the equations

\[
\sin 2\varphi \, |e|^2 \cos 2\varphi - \text{Re}(m\bar{e}) = 0,
\]

\[
\tan^2(\theta/2) = \frac{\text{Re}(m - e \, e^{2i\varphi})}{\text{Re}(m - e \, e^{-2i\varphi})} > 0. \tag{30}
\]

The two pairs of solutions that always exist are on the equator of the sphere (\( \theta = \pi/2 \)) and satisfy

\[
\text{(SS): } \varphi = 0, \pi, \quad \text{(AR): } \varphi = \pi/2, 3\pi/2. \tag{31}
\]

Moreover, if \( |m|^2 > |e|^2 > |\text{Re}(m\bar{e})| \) then there are two additional pairs of solutions satisfying

\[
\text{(SCR): } \cos 2\varphi = \frac{\text{Re}(m\bar{e})}{|e|^2}, \tag{32}
\]

Fig. 4. Fixed points corresponding to the periodic solutions SS, AR, SCR and SR in the spherical system (26) for \( m = n, e = h \). In (a) the SCR solutions do not exist, and in (b) two distinct SCR solutions exist (\(|m|^2 > |e|^2 > |\text{Re}(m\bar{e})|\)).

with \( \tan^2(\theta/2) \) given by (30). One symmetric pair is in the northern hemisphere (\( \tan(\theta/2) < 1 \)) and the other pair is in the southern hemisphere (\( \tan(\theta/2) > 1 \)) (see fig. 4). We now use the implicit function theorem to show that these solutions persist for \((m, e)\) sufficiently close to \((n, h)\).

Specifically, let

\[
g(\theta, \varphi; m, n, e, h) = (g_1(\theta, \varphi; m, n, e, h), g_2(\theta, \varphi; m, n, e, h)), \tag{33}
\]

where SS, AR and SCR satisfy \( g(0, 0; m, n, e, h) = 0 \), and let \( \text{D}g(\theta, \varphi; m, n, e, h) \) be the \( 2 \times 2 \) Jacobian matrix associated with \( g \). Then

\[
\det(\text{D}g(\theta, \varphi; m, n, e, h)) = -2 \sin \theta \{ \cos \theta \sin 2\varphi \text{Im}(m\bar{e}) \\
+ \cos 2\varphi (\text{Re}(m\bar{e}) - |e|^2 \cos 2\varphi) \}. \tag{34}
\]

The determinant (34) is nonzero when evaluated on the SS, AR solutions provided \( |\text{Re}(m\bar{e})| \neq |e|^2 \). Moreover, the determinant is always nonzero when evaluated on the SCR solution since \( |\text{Im}(m\bar{e})| \neq 0 \) follows from the existence condition \( |m|^2 > |e|^2 > |\text{Re}(m\bar{e})| \).

Thus the implicit function theorem applies and the (perturbed) SS, AR and SCR solutions con-
continue to exist on a neighborhood of \((n, h) = (m, e)\). Recall that for the rotating square lattice \((g, f, h) = (c, d, e)\) [9] and for the nonrotating rhombic lattice \((g, f, h) = (d, c, e)\) [10] so that in both of these cases \((n, h) = (m, e)\) on the SB subspace. In each of these cases the existence of SS and AR is guaranteed by the equivariant Hopf theorem. (In the latter case the SS solution corresponds to a standing rectangle pattern.) It follows from these observations that either two or four distinct standing bimodal solutions always bifurcate from the trivial solution if the angle \(\Theta\) is sufficiently close to \(\pi/2\) or if the rotation rate is sufficiently slow.

4. Stability results

In this section we discuss the stability properties of the four primary branches whose existence is guaranteed by the equivariant branching lemma. We also compute the stability properties of the travelling bimodal patterns.

The technique used in establishing the stability of the various patterns has been described in detail by Silber and Knobloch [14]. It relies on the isotropy subgroup of the solution to construct the isotypic decomposition of \(\mathbb{R}^8\) for that solution [11]. In the coordinates suggested by this decomposition the \(8 \times 8\) Jacobian matrix is then block-diagonal. Additional restrictions imposed by the full symmetry \(\Gamma\) can be used to deduce the number of zero eigenvalues and to constrain the structure of the blocks even further. We omit the details and summarize the results in table 3.

The travelling rolls \(TR_1, TR_2\) each have a single zero eigenvalue associated with the phase shift symmetry. The stability with respect to amplitude perturbations is given by \(\text{sgn}(a_r + b_r)\). Stability of \(SR_j\) with respect to \(TR_j\) perturbations is determined by \(\text{sgn}(a_r - b_r)\), as in the Hopf bifurcation with \(O(2)\) symmetry. There are two additional eigenvalues associated with eigenvectors in the SB subspace; the symmetry of the problem forces these eigenvalues to each have multiplicity 2. Note that it is possible for all four of these branches to bifurcate supercritically and yet none be stable.

The possibility of the quasiperiodic (two-frequency) patterns called travelling bimodal patterns was first noted by Knobloch and Silber [9] in their study of the Hopf bifurcation with \(Z_4 \times T^2\) symmetry. These patterns do not exist for all coefficient values. Their stability properties are not hard to establish. We describe here the calculations for \(TB_1\); the results for \(TB_2\) are obtained by interchanging the coefficients \(c\) and \(d\) and the coefficients \(f\) and \(g\).

We determine the linear stability of the \(TB_1\) solution by setting \(z = \left( (\tilde{z}_1 + v_1) e^{i\Omega_1 t}, (\tilde{z}_2 + v_2) e^{i\Omega_2 t}, \tilde{z}_3 e^{i\Omega_3 t}, \tilde{z}_4 e^{i\Omega_4 t} \right)\) in (13), where the \(\tilde{z}_j\) represent perturbations of the \(TB_1\) solution with \(|v_1|, |v_2|, \Omega_1, \Omega_2\) given by (18). We choose the frequencies \(\Omega_3, \Omega_4\) such that \(\Omega_1 - \Omega_2 + \Omega_3 - \Omega_4 = 0\). The stability problem in the \(TB_1\) subspace is determined by setting \(\tilde{z}_3 = \tilde{z}_4 = 0\) and linearizing about \(\tilde{z}_1 = \tilde{z}_2 = 0\). We obtain

\[
\dot{\tilde{z}}_1 = a|v_1|^2 \tilde{z}_1 + av_1 \tilde{z}_2 + cv_1 \bar{v}_2 \tilde{z}_2 + cv_1 v_2 \bar{z}_2,
\]

\[
\dot{\tilde{z}}_2 = a|v_2|^2 \tilde{z}_2 + av_2 \tilde{z}_1 + f v_1 \bar{z}_1 \bar{v}_2 \tilde{z}_2 + f v_1 \bar{v}_2 \tilde{z}_1,
\]

together with their complex conjugates. There are two zero eigenvalues associated with translation symmetry and phase shift symmetry. The remaining two eigenvalues \((s_1, s_2)\) satisfy

\[
s_1 + s_2 = 2a_r(|v_1|^2 + |v_2|^2),
\]

\[
s_1 s_2 = 4(a_r^2 - c_r f_r)|v_1|^2 |v_2|^2.
\]
Table 3
The branching equations and stability assignments for six of the possible patterns. The quantities $m$ and $n$ are defined in (22). A solution is stable if the signed quantities are all negative. A single asterisk indicates a complex conjugate pair of eigenvalues; a double asterisk indicates eigenvalues of multiplicity two. The eigenvalues $s^\pm$ are given in (38); zero eigenvalues are not listed.

<table>
<thead>
<tr>
<th>Name</th>
<th>Branching equation</th>
<th>stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$\lambda + a_i</td>
<td>v_1</td>
</tr>
<tr>
<td>$TR_1$</td>
<td>$\lambda + a_i</td>
<td>v_1</td>
</tr>
<tr>
<td>$TR_2$</td>
<td>$\lambda + a_i</td>
<td>v_1</td>
</tr>
<tr>
<td>$SR_1$</td>
<td>$\lambda + (a_t + b_t)</td>
<td>v_1</td>
</tr>
<tr>
<td>$SR_2$</td>
<td>$\lambda + (a_t + b_t)</td>
<td>v_1</td>
</tr>
<tr>
<td>$TB_1$</td>
<td>$\lambda + a_i</td>
<td>v_1</td>
</tr>
<tr>
<td>$TB_2$</td>
<td>$\lambda + a_i</td>
<td>v_1</td>
</tr>
</tbody>
</table>

The linear stability problem out of this fixed point subspace ($\tilde{z}_1 = \tilde{z}_2 = 0$) is given by

$$\dot{z}_3 = (\mu - i\Omega_3 + b|v_1|^2 + d|v_2|^2)\tilde{z}_3$$
$$+ e\bar{v}_1 v_2 \tilde{z}_4$$

$$\dot{z}_4 = (\mu - i\Omega_4 + b|v_2|^2 + g|v_1|^2)\tilde{z}_4$$
$$+ hv_1 \bar{v}_2 \tilde{z}_3. \tag{37}$$

These equations have time-independent coefficients, and hence the four eigenvalues $s^\pm, s^\pm$ are readily found:

$$s^\pm = \frac{\lambda}{2(a_t^2 - c_t f_t)} \left\{ (a_t - b_t + f_t - g_t)(a_t - c_t) + (a_t - b_t + c_t - d_t)(a_t - f_t) \right\}$$
$$\pm \left\{ \sqrt{((a_t - c_t)(b_t - g_t) - (a_t - f_t)(b_t - d_t)} - \text{im} i_t(a_t - f_t) + \text{im} i_t(a_t - c_t))^2$$
$$+ 4eh(a_t - c_t)(a_t - f_t)^2 \right\}^{1/2} + i\Omega, \tag{38}$$

where $m, n$ are given by (22). Here $\Omega$ is a purely real quantity, and so does not affect the stability properties. Note that the quantity under the square root is complex so that the condition for stability, Re$(s^\pm) < 0$, is no longer simple.

These results have a number of consequences. Since $TB_1$ exist only when $(a_t - c_t)(a_t - f_t) > 0$ and $TB_2$ exist only when $(a_t - d_t)(a_t - g_t) > 0$, it follows that both $TB_1$ and $TB_2$ are present simultaneously only when $TR_1$ and $TR_2$ have identical stability properties. Moreover if $a_t - c_t > 0$, then $TB_1$ is supercritical only if $a_t - c_t > 0$, in which case it cannot be stable, and similarly for $TB_2$. On the other hand if $a_t - c_t < 0$, then $TB_1$ are supercritical only if $a_t^2 - c_t f_t < 0$, in which case their stability depends on the eigenvalues $s^\pm$ describing the perturbations out of the $TB_1$ fixed point subspace. In the simpler case of the nonrotating rhombic lattice the TB states are periodic ($\Omega_1 = \Omega_2$) and correspond to travelling rectangle solutions. Since in this case these states can be stable [10], it follows by continuity of the eigenvalues that the TB states on the rotating rhombic lattice can also be stable, at least for sufficiently slow rotation rates.

We observe, finally, that the SS and AR states can both be stable for a range of coefficient
values on the rotating square lattice [9] and on the nonrotating rhombic lattice [10], where their existence is guaranteed by the equivariant Hopf theorem. Moreover, it is possible for both of them to be stable simultaneously or for one of them to coexist stably with the SR solutions [9,10]. Both AR and SS have three zero eigenvalues, associated with translations in two independent directions and the phase-shift symmetry; generically the remaining eigenvalues have nonzero real part. When the discrete symmetry is reduced from $Z_4$ (or $D_2$) to $Z_2$ the zero eigenvalues remain while the real parts of the other eigenvalues stay bounded away from zero provided the symmetry breaking from $Z_4$ (or $D_2$) to $Z_2$ is sufficiently weak. Since the former is the limit of the rhombic lattice as $\Theta \to \pi/2$ while the latter is its limit for slow rotation rates it follows that $Z_2$ symmetric analogues of the SS and AR states persist on the rotating rhombic lattice and that one or both may be stable. On the other hand the SCR states (32) are saddles in the associated spherical system (26) when $(n,h) = (m,e)$ [16]; we expect them to remain unstable for coefficient values close to $(n,h) = (m,e)$.

5. Heteroclinic cycles and quasiperiodic standing waves

In this section, we focus on instabilities of the TR and SR solutions in the TB and SB subspaces, respectively. We determine conditions for the formation of various types of heteroclinic orbits connecting either the two travelling rolls states or the two standing rolls states. We associate the formation of such heteroclinic orbits with the onset of a Küppers–Lortz-like instability for oscillatory rotating convection.

We begin by recalling that if the TR and SR solutions both bifurcate supercritically, then within the MR subspace one of them is stable while the other is unstable. We consider first the case where the TR are supercritical ($a_t < 0$) and stable with respect to counterpropagating perturbations: $b_t - a_t < 0$. In order to understand the origin of the Küppers–Lortz (hereafter KL) instability for travelling rolls, we consider the case where the TR states are stable for small enough rotation rates. Increasing the rotation rate can then trigger the KL instability in one of four ways, associated with the four ways that the quasiperiodic TBi or TBz solutions may cease to exist as $(c_t - a_t)(f_t - a_t)$ or $(d_t - a_t)(g_t - a_t)$ change from positive to negative. To see this consider the dynamics in the TBi subspace when $(c_t - a_t)(f_t - a_t) < 0$. Writing $z = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, 0, 0)$ in equation (13) one finds that the amplitude dynamics $(i_1, i_2)$ decouple from the phase dynamics $(\phi_1, \phi_2)$, and

$$\dot{r}_1 = (\lambda + a_t r_1^2 + c_t r_2^2) r_1$$
$$\dot{r}_2 = (\lambda + a_t r_2^2 + f_t r_1^2) r_2.$$  (39)

The TR$_1$ solution satisfies $(r_1, r_2) = (r, 0)$ and the TR$_2$ solution satisfies $(r_1, r_2) = (0, r), r^2 = -\lambda/a_t$. A sufficient condition for trajectories to remain bounded in the $(r_1, r_2)$ plane is $c_t + f_t + 2a_t < 0$, in which case there is a saddle-sink heteroclinic orbit between TR$_1$ and TR$_2$ in the $(r_1, r_2)$ plane [17]. If $f_t - a_t > 0$ then the orbit originates at TR$_1$ and terminates at TR$_2$ in the TB$_1$ subspace. In this case the KL instability is in the counterclockwise or prograde direction. If instead $c_t - a_t > 0$ then TR$_2$ lose stability to TR$_1$ in the TB$_1$ subspace and the KL instability is in the clockwise or retrograde direction. Alternatively, the KL instability may occur in the TB$_2$ subspace when $(d_t - a_t)(g_t - a_t) < 0$. If $d_t - a_t > 0$ then the instability is from TR$_2$ to TR$_1$ in the prograde direction, whereas if $g_t - a_t > 0$ then the KL instability is in the opposite direction. In all of these cases the instability is completely analogous to the KL instability for steady rolls (see section 1). As in the steady case once a TR loses stability to another TR at an angle $\Theta$ to itself the new pattern is again unstable to like perturbations at the same angle $\Theta$ to itself, and so on. Note that as for the KL instability of steady
rolls the KL instability of travelling rolls is associated with the disappearance of an unstable pattern – in this case a TB pattern.

An interesting situation arises when both TR solutions are stable in the MR subspaces \((b_r < a_r < 0)\), but unstable in the full space \((f_r - a_r)(g_r - a_r) < 0, (c_r - a_r)(d_r - a_r) < 0\). This situation may occur at fixed \(\Theta\) for rotation rates larger than the critical rotation rate where the KL instability first appears. In this case, we expect a continuous band of angles \(\Theta\) for which TR are KL unstable. The heteroclinic cycle we now describe will be present provided the band includes angles \(\Theta\) and \(\pi - \Theta\) (e.g., \(\Theta \approx \pi/2\)). Note first that the absence of the quasiperiodic travelling bimodal solution in the TB₁ subspace \((f_r - a_r)(c_r - a_r) < 0\) now implies the absence of the quasiperiodic TB₂ solution as well \((d_r - a_r)(g_r - a_r) < 0\). In this case when \(f_r - a_r > 0\) there is a structurally stable heteroclinic orbit connecting TR₁ to TR₂ in the TB₁ subspace at the same time as there is a similar orbit connecting TR₂ to TR₁ in the TB₂ subspace, provided the trajectories remain bounded in both subspaces. By applying the discrete symmetries of the rhombic lattice it is easy to see that under these conditions the above heteroclinic orbits are in fact parts of a structurally stable heteroclinic cycle connecting, in a counterclockwise fashion, four sets of travelling rolls in different directions (see fig. 5). The structural stability of the cycle, which is composed of saddle–saddle connections in the full eight-dimensional space, is a consequence of the symmetries that guarantee the existence of the invariant fixed point subspaces [18]. Sufficient conditions for the existence of such a heteroclinic cycle are

\[
c_r + f_r + 2a_r < 0, \quad d_r + g_r + 2a_r < 0; \quad (40)
\]
\[
f_r - a_r > 0, \quad g_r - a_r < 0,
\]
\[
c_r - a_r < 0, \quad d_r - a_r > 0, \quad (41)
\]

where conditions (40) guarantee that trajectories remain bounded in the TB subspaces. Moreover, with the help of theorem 2.10 of Melbourne et al. [17] we can show that the following conditions suffice for the cycle to be asymptotically stable:

\[
\min(-2a_r, a_r - g_r, f_r - b_r) \times \min(-2a_r, a_r - c_r, d_r - b_r) > (f_r - a_r)(d_r - a) > 0 > a_r. \quad (42)
\]

Fig. 5. The structurally stable attracting heteroclinic cycle shown in fig. 6 connects four travelling roll solutions as shown above. The TR₃, j = 1, 2, state is labeled by its \((z_1, z_2, z_3, z_4)\) coordinates. The superscripts \(t\) and \(r\) indicate whether the rolls are left-travelling or right-travelling.

Fig. 6 illustrates the approach to such a cycle. It is characterized by an abrupt cyclic switching between the four sets of travelling rolls, with the length of time it takes to complete the full cycle increasing each time around. Note that the heteroclinic cycle is traversed in the opposite direction if the inequalities (41) are all reversed (i.e., \((f_r, g_r) \leftrightarrow (c_r, d_r)\)). The heteroclinic cycle described here generalizes the cycle present on the rotating square lattice [9]. However, in the square case the four travelling rolls are related by symmetry (rotations by \(\pi/2\)) and the formation of the four way heteroclinic cycle is inevitable whenever there is a KL instability for \(\Theta = \pi/2\).

The possibilities for a KL instability of standing rolls are much richer. We take \(a_r < -|b_r|\) so that SR are supercritical and stable with respect to travelling roll perturbations. In this case, instabilities of the SR solutions are investigated
in the SB subspace. This is accomplished by examining the dynamics on the sphere (26). To guarantee that the trajectories on $S^2$ represent bounded dynamics in $\mathbb{R}^3$, we require $g_0(\theta, \varphi) < 0$ in (24) for all $\theta, \varphi \in [0, \pi]$; a sufficient condition for this inequality to hold is $4(a_r + b_r) < m_r + n_r - |e| - |h|$, where $m$ and $n$ are given by (22). In this case, orbits of (24) are attracted to a two-dimensional invariant manifold in $\mathbb{R}^3$; (26) then represents a projection of the dynamics onto the unit sphere [16]. In this way, we are guaranteed that fixed points, limit cycles and heteroclinic orbits in the two-dimensional spherical system (26) represent the same types of structures in the $\mathbb{R}^3$ system (24). Recall that a fixed point in the spherical system represents a periodic orbit in the SB subspace. Similarly, a limit cycle in the spherical system corresponds to a quasiperiodic solution of (21).

We now discuss the possible asymptotic dynamics when one of the SR solutions is stable and the other is unstable in the SB subspace. Specifically, we assume that SR\(_2\) is the stable solution ($m_r > 0, |m|^2 > |e|^2$). We restrict our discussion, for the time being, to the dynamics of the spherical system (26). There are two subcases to consider: (1) SR\(_1\) is a saddle ($|n|^2 < |h|^2$), (2) SR\(_1\) is a source ($n_r < 0, |n|^2 > |h|^2$). In the case where SR\(_1\) is a saddle, we expect submaximal fixed points to exist in the spherical system. If these are unstable and there are no limit cycles present in the spherical system, then, generically, the unstable manifold of SR\(_1\) lies in the stable manifold of SR\(_2\) and we have a heteroclinic orbit from SR\(_1\) to SR\(_2\). In the case where SR\(_1\) is a source, we can readily determine sufficient conditions for the stable manifold of SR\(_2\) to coincide with the unstable manifold of SR\(_1\).
in the spherical system. This is accomplished by demanding \( g_1(\theta, \varphi) > 0 \) in (26) for all \( \theta, \varphi \in [0, \pi] \) so that \( \theta' > 0 \) everywhere except at \( \text{SR}_1 \) and \( \text{SR}_2 \) (i.e., at \( \theta = 0, \pi \)). We observe that \( g_1 > 0 \) if \( m, > |e| \) and \( n, < -|h| \) (fig. 3a). In this case, there do not exist any quasiperiodic or submaximal periodic solutions in the SB subspace, and \( \text{SR}_2 \) is a global attractor in the SB subspace.

The possibilities for heteroclinic connections between \( \text{SR}_1 \) and \( \text{SR}_2 \) described above were identified by Swift and Barany [111] in the special case of standing rolls on a hexagonal lattice. In this case, there are three sets of standing rolls (\( \text{SR}_1, \text{SR}_2, \text{SR}_3 \)), each with a different wave vector orientation. The rolls, related by \( 2\pi/3 \) rotations of the hexagonal lattice, have the same stability properties. Consequently, if there is a heteroclinic connection from \( \text{SR}_1 \) to \( \text{SR}_2 \), then there is also one from \( \text{SR}_2 \) to \( \text{SR}_3 \) and from \( \text{SR}_3 \) back again to \( \text{SR}_1 \). Thus the onset of the KL instability for \( \text{SR} \) on the hexagonal lattice is associated with the formation of a heteroclinic cycle. Moreover, Swift and Barany showed that when the cycle consists of saddle–focus connections in the spherical system there is the possibility of chaotic dynamics of Shil’nikov type.

Finally, we point out that it is possible for \( \text{SR}_1 \) and \( \text{SR}_2 \) to be the only fixed points in the spherical system and for both of them to be unstable. The Poincaré–Bendixson theorem [19] then guarantees the existence of a limit cycle in the spherical system; this corresponds to a quasiperiodic standing wave solution of (13). Sufficient conditions for a limit cycle to exist in the spherical system (26) are

\[
\begin{align*}
\varphi' &< 0, \quad n, < 0, \quad |m| > |e|, \quad |n| > |h|; \\
m_i &< -|e|, \quad n_i > |h|,
\end{align*}
\]

or \( m_i > |e|, \quad n_i < -|h| \). (44)

The inequalities (43) guarantee that \( \text{SR}_1 \) and \( \text{SR}_2 \) are both sources in the spherical system. The inequalities (44) guarantee that \( \varphi' > 0 \) or \( \varphi' < 0 \), respectively, everywhere on the sphere. Thus the only fixed points on the sphere are at the north pole and the south pole (i.e., \( \text{SR}_1 \) and \( \text{SR}_2 \)). Since the dynamics are bounded on \( S^2 \), and the only fixed points are sources at the poles, there must exist a limit cycle encircling the sphere. Fig. 7 shows the approach to such a quasiperiodic solution of (13). Note that the quasiperiodic standing bimodal pattern is not a relative equilibrium (i.e., it does not coincide with the group orbit of a point in \( \mathbb{C}^4 \)). In this way it differs fundamentally from the quasiperiodic travelling bimodal patterns: there does not exist a co-moving frame in which the quasiperiodic SB solutions are singly-periodic. As with the TB states, the quasiperiodic SB solution may appear in a primary bifurcation. Furthermore, we note that the quasiperiodic SB solution may vanish in a Hopf bifurcation with, say, \( \text{SR}_2 \) as \( n, \) is varied through 0 leaving in its wake a KL instability of \( \text{SR}_1 \) to \( \text{SR}_2 \).

6. Discussion and conclusion

In this paper we have studied doubly-periodic patterns on the rotating rhombic lattice. We have shown that there are always four primary branches of (temporally) periodic states: these are the travelling and standing rolls in the two directions \( k_1 \) and \( k_2 \). In addition we showed that in open regions in coefficient space there may be two or four additional primary branches of periodic solutions with submaximal isotropy; in particular these exist for a nearly square rotating lattice or for a rhombic lattice that rotates slowly. We also found two primary branches of two-frequency (quasiperiodic) travelling states in open regions in coefficient space. The existence of an additional primary bifurcation, to a quasiperiodic standing wave pattern, was inferred using the Poincaré–Bendixson theorem. We have not, however, determined whether any nonsymmetric triply periodic solutions (QP3) exist as primary branches. Our results are closely related to existing results for the rotating square...
lattice [9] and the nonrotating rhombic lattice [10]. We highlight here some of the key differences.

On the rotating rhombic lattice there are two branches of travelling rolls and two of standing rolls, each with different stability properties. On the square lattice the TR branches are identified by an additional symmetry (rotation through $\pi/2$), as are the two SR branches. On the nonrotating rhombic lattice these solution branches are related by reflection symmetries. In both of these special cases the stability properties of the TR solutions are identical, as are the stability properties of the SR solutions. As discussed in section 3, there are two additional patterns with maximal isotropy on both the rotating square lattice and the nonrotating rhombic lattice: these correspond to primary branches of standing squares (SS) (or standing rectangles in the rhombic case) and of alternating rolls (AR). On the rotating rhombic lattice these become standing bimodal patterns with submaximal isotropy and are no longer always present. The two quasiperiodic travelling bimodal patterns on the rotating rhombic lattice are related by a $\pi/2$ rotation symmetry on the rotating square lattice and thus have the same branching and stability properties. On the nonrotating rhombic lattice, the two travelling bimodal solutions become single frequency travelling rectangle patterns with maximal isotropy; the patterns travel along the two axes of reflection symmetry of the rectangle and do not have the same stability properties.

In the previous section we identified various types of Küppers–Lortz instabilities for both travelling and standing rolls. We use this term to refer to instabilities of a roll pattern with respect to another pattern with a different orientation. These instabilities do not occur in nonrotating systems for which reflection symmetry precludes a preferred direction of instability. In the case of travelling rolls the instability is of precisely the same form as the well-known Küppers–Lortz instability for steady rolls; in both cases the instability is triggered by the disappearance of an unstable pattern and the formation of a heteroclinic orbit. Under appropriate conditions we found, in addition, a structurally stable heteroclinic cycle connecting rolls travelling in four different directions. This cycle is the direct analogue of the structurally stable heteroclinic cycle whose formation triggers the KL instability of travelling rolls on the rotating square lattice. On the rotating rhombic lattice the cycle exists if the fluid rotates rapidly enough for such an instability to be present in a band containing both $\Theta$ and $\pi - \Theta$, where $\Theta$ is the angle between $k_1$ and $k_2$. While this cycle may be asymptotically
stable when the solution space is restricted to patterns periodic on the rhombic lattice, it is not attracting in the infinite-dimensional hydrodynamic problem. This is because each TR solution is unstable to a continuum of TR oriented at all angles $\Theta$ within the KL unstable band. The resulting dynamics in the infinite-dimensional case may be quite complicated.

For travelling rolls we found that whether or not a KL instability occurs is largely determined by the linear stability of the TR states. This is not the case for SR solutions. We did not derive a simple criterion, in terms of the coefficients of the normal form equations, for existence of a KL instability for SR. The difficulty arises in trying to rule out the possible coexistence of quasiperiodic or submaximal periodic attractors in the SB subspace with a stable SR solution when the other SR state is unstable. We were able to do this only in a particular case where one of the SR solutions was a sink and the other a source in the associated spherical system. In this case we could identify the origin of the KL instability with the disappearance of a quasiperiodic standing wave solution.

Thus far we have focused attention on the general rhombic lattice, and treated the square lattice as a special case. The square lattice has the property that a KL instability of travelling rolls guarantees the existence of a heteroclinic cycle connecting the four TR states. This is because the four competing TR patterns are related by symmetry (rotation by $\pi/2$). In contrast, the observation that the two orthogonal SR patterns must have the same stability properties may be used to prove that there cannot be a KL instability for SR at $\Theta = \pi/2$. Specifically, on the rotating square lattice the only possibility for a heteroclinic connection between the two SR solutions is via a structurally unstable saddle–saddle connection in the spherical system (26). Thus there is no KL instability of standing rolls on the square lattice. There is one other doubly-periodic lattice that has the property that the SR and TR states are related by a rotational symmetry of the lattice. This is the hexagonal case, studied extensively in the nonrotating case for both steady state [13] and Hopf bifurcations [20]. It has also been analyzed for steady state bifurcations in the rotating case [15]. Some aspects of the dynamics on the hexagonal lattice follow as special cases of the analysis presented above for the rhombic lattice. We deduce these below but make no attempt to provide a complete discussion of this problem. We begin by observing that the stability of TR and SR on the hexagonal lattice may be determined by considering the special case of the rhombic lattice with $\Theta = \pi/3$. We let $\Gamma_h \equiv \mathbb{Z}_6 \times T^2 \times S^1$ denote the symmetry group associated with the hexagonal lattice. In this case there are three left-travelling rolls in the $k_1, k_2, k_3$ directions with amplitudes $z_1, z_2, z_3 \in \mathbb{C}$, and three right-travelling rolls with amplitudes $w_1, w_2, w_3 \in \mathbb{C}$. Here we take $|k_1| = |k_2| = |k_3| = k_c$ and $k_1 + k_2 + k_3 = 0$. The action of $\Gamma_h$ on $\mathbb{C}^6$ is generated by

\begin{align}
\rho: \quad & (z_1, z_2, z_3; w_1, w_2, w_3) \\
& \rightarrow (z_2, z_3, z_1; w_2, w_1, w_3), \\
\kappa: \quad & (z_1, z_2, z_3; w_1, w_2, w_3) \\
& \rightarrow (w_1, w_2, w_3; z_1, z_2, z_3), \\
(\theta_1, \theta_2): \quad & (z_1, z_2, z_3; w_1, w_2, w_3) \\
& \rightarrow (e^{i\theta_1} z_1, e^{-i(\theta_1 + \theta_2)} z_2, e^{i\theta_2} z_3; \\
& e^{-i\theta_1} w_1, e^{i(\theta_1 + \theta_2)} w_2, e^{-i\theta_2} w_3), \\
\phi: \quad & (z_1, z_2, z_3; w_1, w_2, w_3) \\
& \rightarrow e^{i\phi} (z_1, z_2, z_3; w_1, w_2, w_3),
\end{align}

where $\rho, \kappa \in \mathbb{Z}_6$, $(\theta_1, \theta_2) \in T^2$, and $\phi \in S^1$. The two rhombic subspaces containing the SR$_1$ solution are RH$_1$: $(z_1, z_2, 0; w_1, w_2, 0)$ and RH$_2$: $(z_1, 0, z_3; w_1, 0, w_3)$. These are related by the rotational symmetry $\rho \in \mathbb{Z}_6$. Thus we focus on the invariant subspace with $z_3 = w_3 = 0$ ($\Theta = \pi/3$), which is fixed by the group element $[(\pi, 0), \pi] \in T^2 \times S^1$. We note that the group
elements of $\Gamma_h$ act on points in the rhombic subspaces in precisely the same manner as $\Gamma$ acts on $C^4$. Thus, the $\Gamma_h$-equivariant vector field, restricted to $R_{H_1}$ is $\Gamma$-equivariant. That this restriction yields a generic $\Gamma$-equivariant vector field may be verified for the cubic truncation of the general $\Gamma_h$-equivariant vector field [11]:

$$
\dot{z}_1 = z_1 (\mu + a|z_1|^2 + b|w_1|^2 + c_1|z_2|^2 + d_1|w_2|^2 + c_2|z_3|^2 + d_2|w_3|^2) + \bar{w}_1 (e_1 z_2 w_2 + e_2 z_3 w_3),
$$

where the remaining components of the vector field are generated by letting $Z_6$ act on $\dot{z}_1$ and the coefficients $\mu, a, b, c_j, d_j, e_j$ ($j = 1, 2$) are all complex. The stability of TR and SR on the hexagonal lattice may be determined by calculating the stability of $TR_j, SR_j$ ($j = 1, 2$) in the rhombic subspace $R_{H_1}$. For SR there are four eigenvalues, two of which are zero, in the MR subspace $(z_1, 0, 0; w_1, 0, 0)$. The remaining eight eigenvalues are associated with the SB subspace of $R_{H_1}$: two eigenvalues, each of multiplicity two, for each of $SR_1$ and $SR_2$. For TR there are also four eigenvalues associated with the MR subspace: one is zero, one is associated with the branching direction and two come as a complex conjugate pair. The remaining eight eigenvalues are associated with the TB$_1$ and TB$_2$ subspaces of $R_{H_1}$: two complex conjugate pairs for each of $TR_1$ and $TR_2$. Note that on the hexagonal lattice the six left- and right-travelling rolls in the directions $k_j$ ($j = 1, 2, 3$) are related by symmetry as are the three standing rolls. Thus they must all have the same stability properties.

Although the calculations described in this paper are completely general they are motivated by convection in a plane layer rotating about the vertical. It was shown by Chandrasekhar [4] that in such a layer convection sets in as overstable oscillations for sufficiently small Prandtl numbers Pr. When the centrifugal force is small relative to gravity the rotation affects the system through the Coriolis force only, and the resulting equations for an unbounded layer are translation-invariant. With the assumptions discussed in section 1 the governing Navier–Stokes equations can be reduced to the normal form (6) if the onset of convection is steady or to the normal form (13) if it is oscillatory. In the former case the KL instability is triggered when $c - a$ becomes positive. For large Pr this first occurs for an angle $\Theta$ near $59^\circ$ and the resulting instability is in the direction of rotation [3,5]. For the oscillatory instability states only partial results are available. Knobloch and Silber [21] showed that with stress-free and fixed temperature boundary conditions at the top and bottom TR were preferred over SR in the range $0.53 < Pr < 0.68$ with SR preferred for small rotation rates and TR preferred for large rotation rates when $Pr < 0.53$. Qualitatively similar results hold also for rigid boundaries [22]. However, in these papers the stability properties of these states with respect to perturbations oriented at some angle $\Theta$ were not investigated. More recently, Riahi [23] has considered the stability properties of standing patterns with stress-free boundary conditions with respect to perturbations in various directions. He finds that standing rolls are always supercritical and always unstable with respect to such perturbations (provided $\Theta \neq \pi/2$ for reasons explained above). Riahi thus shows that the standing rolls that are stable with respect to travelling roll perturbations with $\Theta = 0$ are always unstable in the SB subspaces. However, the dynamics in the SB subspace was not determined. In particular, Riahi did not consider standing wave patterns in which the two sets of standing rolls were phase shifted, in time, relative to each other. We are thus unable to determine from Riahi's calculation whether the instability of the standing rolls is in fact a Küppers–Lortz type of instability. However, even if this is the case, it is likely to be a consequence of the stress-free boundary conditions used; with these boundary conditions steady rolls are also always KL unstable [24]. The techniques used in this paper clarify the mechanism for KL-like instabilities, and
in addition specify precisely the calculations that
must be performed to establish their existence.
We hope that this paper will spur efforts in this
direction.

Acknowledgements

This work was presented at a conference on Bi-
furcation and Symmetry, held in Marburg, Ger-
many, in June 1991. We have benefited from
discussions with Ian Melbourne and Jim Swift.
The work of E.K. was supported in part by NSF
grant DMS-8814702 as well as an INCOR grant
through the Center for Nonlinear Science at Los
Alamos National Laboratory.

References

[1] M. Golubitsky, I. Stewart and D.G. Schaeffer,
Singularities and Groups in Bifurcation Theory, Vol. II,
vol. 69 of Springer Series in Appl. Math. Sci. (Springer,
New York, 1988).
in Physical Systems – Pattern Formation, Chaos and
Waves, eds. L.Lam and H.C. Morris (Springer, New
[8] F. Zhong, R. Ecke and V. Steinberg, Physica D 51
(1991) 596.
[9] E. Knobloch and M. Silber, in: Bifurcation and
Symmetry: Cross Influences between Mathematics
and Applications, eds. E. Allgower, K. Böhmer and
[10] M. Silber, H. Riecke and L. Kramer, Symmetry-
breaking Hopf bifurcation in anisotropic systems,
1063.
selection in rotating convection: hexagonal symmetry,
Phil. Soc. 103 (1988) 189.
Systems and Chaos, vol. 2 of Texts in Applied
36 (1986) 283.
Dyn. 51 (1990) 195.
[22] T. Clune and E. Knobloch, Pattern selection in rotating
convection with experimental boundary conditions,