Nonlinear Dynamics and Chaos:  
Where do we go from here?

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Preface

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Chapter 1

Outstanding Problems in the Theory of Pattern Formation

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1.1 Introduction

Many systems of interest in physics, chemistry and the biological sciences exhibit spontaneous symmetry-breaking instabilities that form structures that we may call patterns [34, 35]. The appearance of convection cells in a fluid layer heated from below [81], or of vortex structures in the flow between two independently rotating cylinders [81, 55], are familiar examples from fluid mechanics. But there are many other systems that form patterns. Recently studied examples of spatially periodic structures include the Turing instability [99], and the Faraday system [89, 91]. Vertically vibrated granular media exhibit similar pattern-forming instabilities [96, 102]. Related instabilities have been identified in the primary visual cortex and may be responsible for hallucinations (P C Bressloff and J D Cowan, this volume). Other systems form spiral waves or target patterns, emanating from apparently random locations, perhaps triggered by impurities or specks of dust [1]. A remarkable recent discovery is that of oscillons, or localized oscillations, in vertically vibrated granular media [157]. These oscillons may form ‘bound states’ that have been called dimers. Certain chemical systems break up into spots which grow, and fission into new spots or disappear depending on the density of spots around them [100, 83]. Such pattern-forming systems thus behave almost like a colony of living organisms.

The problems these experiments and simulations raise are fundamental. These range from the particular to the general. Typical of the former are studies of specific systems, which attempt to identify pattern-forming instabilities and their subsequent evolution using the (known) field equations describing the system. This approach may use analytical techniques
(stability theory, perturbation methods) but often revolves around direct numerical simulation of the system. Significantly, the experiments and simulations often show that different systems behave in an almost identical fashion, regardless of the specific equations governing the system and the physical mechanism behind the pattern-forming instability. For this reason I focus here on the universal aspects of pattern formation - those aspects that are independent of the details of the governing equations. Indeed, even when the governing field equations are known, detailed quantitative comparisons with experiments have only rarely been attempted. Often a theory is only tested against measurements of linear stability thresholds. Such tests, while valuable, cannot be taken as confirming the theory. In general it is difficult to integrate the three-dimensional field equations describing a system, particularly if it is spatially extended. In such circumstances one resorts to simplifying assumptions which are believed to retain the essence of the system. These simplified formulations yield a great deal of valuable information, but often preclude detailed quantitative comparisons. This is regrettable, since a qualitative agreement between experiment and theory may in fact be fortuitous. The recent study of binary fluid convection by Batiste et al [121] shows just how difficult it is to perform quantitative comparisons even with very carefully documented experiments.

In the remainder of this section I mention a number of outstanding issues in the theory of pattern formation that are of particular personal interest. The list is not intended to be exhaustive, but rather to stimulate interest in certain topics, some longstanding, some new, where progress is needed, and to summarize both the state of the field and formulate the right questions. Some of these are then explored in greater detail in Sections 1.2-6.

For simplicity the discussion is divided into two subsections, loosely called ‘weakly nonlinear theory’ and ‘fully nonlinear theory’. To some extent this is an artificial division, and there is much overlap between the two sets of topics. A relatively recent review of the subject may be found in Ref. [59].

**Weakly nonlinear theory**

- Applicability and rigorous justification of envelope equations
- Boundary conditions for envelope equations
- Pattern formation via nonlocal envelope equations
- Dynamics of weakly damped driven systems
- Applicability of integrable PDEs to physical systems
- Quasi-Patterns

Within weakly nonlinear theory we know very well how to deal with bounded systems undergoing a pattern-forming instability. Such systems
have a spectral gap between any marginally unstable eigenvalue, and the remaining stable eigenvalues. When there are no zero eigenvalues related to continuous symmetries such as rotational invariance centre manifold reduction shows that the dynamics of the stable modes are slaved to the slow evolution of the near marginal mode (or modes). A systematic procedure has been developed for dealing with problems of this type. First, the critical parameter value(s) is (are) identified at which one or more modes are neutrally stable; these modes have eigenvalues on the imaginary axis and are called centre modes. Centre manifold reduction is performed on the system (ODEs or PDEs) to obtain the equations governing the (slow) evolution of the system on the centre manifold. Since the centre manifold is in general low-dimensional this procedure amounts to a dramatic reduction in the number of degrees of freedom of the system. Next, near-identity nonlinear coordinate changes are performed to put the resulting equations into a simple form, called a normal form, and are chosen to preserve the dynamics of the system. Consequently the stability properties of the various fixed points and limit cycles are unaffected by this simplification. The structure of the resulting normal form depends only on the linear problem and any symmetries (or degeneracies) that may also be present. Hence the same normal form finds application in a large variety of circumstances. These normal forms must then be ‘unfolded’ by including the terms generated by small changes of the parameter(s) away from critical. In many cases these normal forms and their unfoldings have been worked out, and their dynamics are well understood. Different systems that reduce to the same normal form differ only in the values of the coefficients of the nonlinear terms in these equations. Separate techniques exist for the computation of these coefficients, and typically employ symbolic manipulation routines. It is sufficient usually to compute these coefficients for the critical parameter value(s). This approach has seen (and will continue to see) a large number of applications [58]. The main theoretical issue which remains unsolved concerns the truncation of the normal forms and conditions for the adequacy of the suggested unfoldings. In practice one does not know whether the complete (local) behaviour of the system is captured by any finite truncation of the normal form, or how many unfolding parameters are required. In other words, one does not know whether the codimension of the problem is finite. These questions have only been answered for steady state bifurcation problems where a general procedure for answering these questions exists [77], at least in simple cases. Most problems, however, involve dynamics and it appears unlikely that a corresponding theory can be developed for such cases. In practice the pragmatic approach of truncating the unfolded normal form at various orders and comparing the results works well. Often the inclusion of higher order terms and/or the full parameter dependence of the nonlinear terms enlarges the range of applicability (in parameter space) of the theory, sometimes dramatically [158].
Liapunov-Schmidt reduction [77] can be used to establish rigorous results about solutions of simple type (steady states, relative equilibria or periodic orbits) even when other more complicated solutions are present.

Symmetry, if present, usually plays a paradoxical role. On the one hand it may increase the (geometric) multiplicity of the critical eigenvalue, leading to a higher-dimensional centre manifold and typically multiple solution branches. In particular if the centre manifold is at least three-dimensional complex dynamics may be present near the bifurcation [4]. On the other hand the presence of the symmetry can be exploited to solve these higher-dimensional equations, and hence allows one to understand behaviour that would be difficult to understand in generic systems [45]. For example, the action of the symmetry on an unstable eigenfunction may generate a linearly independent unstable eigenfunction with the same eigenvalue. This is the case if the instability breaks the symmetry of the problem and the symmetry acts absolutely irreducibly on the space of eigenfunctions. In contrast, continuous symmetries such as SO(2) which do not act absolutely irreducibly do not generically admit steady state symmetry-breaking bifurcations at all, and the associated zero eigenvalue due to rotation invariance is then responsible for phase drift, i.e., the instability takes the form of a wave. However, at the same time the SO(2) invariance of the system implies that the spatial phase of the wave is decoupled from its amplitude.

In unbounded systems the situation is much less well understood. Here there is usually no spectral gap and the stable eigenvalues accumulate on the imaginary axis. In addition in two or more spatial dimensions there is often an orientational degeneracy. This is so, for example, in spatially isotropic systems. In such systems the linear stability theory predicts the wave-number of the marginally stable modes but not their direction. Both problems can be traced to the presence of a non-compact symmetry group, the Euclidean group in two or three dimensions. Two techniques have been developed to deal with these problems. Envelope equations employ a slowly spatially varying envelope to take into account wave-vectors near a particular marginal wave-vector, but because they focus on the behaviour of modes of a certain type they are unable to capture significant changes in the magnitude and direction of the dominant wave-vector. The derivation of the corresponding equations is formal, and until recently there were no rigorous results about the validity of the commonly used envelope equations. In one dimension the first such results were given by van Harten [116] and Schneider [109]. These authors show that solutions of the Ginzburg-Landau equation track solutions of the original PDE on $O(\epsilon^{-1})$ length scales for $O(\epsilon^{-2})$ times. The most complete results on the validity and universality of the Ginzburg-Landau equation are given by Melbourne [87, 88] who discusses in detail the significance of higher order terms that depend on the fast spatial variable, and the way these terms lock the envelope to the
underlying pattern.

Both the above problems can be avoided by posing the PDE on a lattice, i.e., by imposing periodic boundary conditions in the unbounded directions. The translations then form a compact symmetry group (circle or torus group), and the allowed wave-vectors become discrete. Problems of this type are amenable to straightforward analysis, and have now been worked out for both steady and Hopf bifurcations in two, and in some cases, three dimensions. This approach has had a number of successes, both in the theory of convection [128] and in the Faraday system [44, 36, 42]. The main reason is that the amplitude equations are easy to generate using group-theoretic techniques that utilize the symmetry properties of the chosen lattice, and that their solutions can be analyzed and their stability properties determined in the abstract. Even without coefficient calculation, the results from this approach often provide a qualitative understanding of the behaviour of the system. For some of the higher-dimensional representations the corresponding analysis may be quite involved and techniques from computer algebra (Gröbner bases etc) prove invaluable [123]. A brief review of these techniques is provided in Section 2. Although there are still a number of cases that have not been worked out, the main theoretical interest undoubtedly centres on finding ways to relax the periodic boundary conditions and thereby make the analysis applicable to a broader class of problems. Attempts to do this include extensions of the Newell-Whitehead formalism to two and three-dimensional patterns [38, 39, 40, 41] but are neither rigorous nor entirely convincing.

Although the use of envelope or amplitude equations is common they are typically used as model equations. This is the case when the equations are not derived via a rational expansion and hence still depend on the expansion parameter; in other cases the small parameter is retained in the boundary conditions imposed on the envelope equation. However, it is even more common to replace the boundary conditions by periodic ones, and to argue that the resulting solutions apply to systems in large domains. This approach is in general dangerous, and is no substitute for deriving the correct boundary conditions at the same time as the envelope equation. Unfortunately this often entails matching the solutions of the envelope equation to boundary layers computed from the original field equations, and is quite involved. In fact, the correct boundary conditions for the Hopf bifurcation in extended domains, hitherto treated heuristically [16], have only been derived recently [14, 15], while those for the steady state bifurcation are discussed in [17].

In many cases the formal asymptotic methods lead to nonlocal amplitude or envelope equations. This is typically a consequence of the presence of some constraints such as mass conservation, or the presence of multiple time scales. In catalysis pressure effects are often responsible for the presence of nonlocal (global) coupling [32, 28]. Charge density waves in n-type
semiconductors produced by the Gunn instability are described by a local equation subject to a nonlocal constraint due to an imposed voltage bias across the system [33]. In other cases nonlocal terms are present due to the excitation of near marginal large scale modes, such as mean flows. These cases are examined in more detail in Section 1.3.

A great deal of effort has been devoted to the derivation of envelope or long-wave equations for ideal (i.e., non-dissipative) systems, such as ideal fluid flow. Water wave theory has led to several much studied equations of this type, the nonlinear Schrödinger equation, and the Korteweg-de Vries and Davey-Stewartson equations. All three are examples of \textit{integrable} equations, and admit localized structures called solitons that interact in a particle-like manner. However, real fluids are dissipative and the solitons must be sustained against dissipation by weak forcing. For example, in optical transmission lines solitons require periodic (spatial) pumping to restore their amplitude and compensate losses. The resulting system is no longer completely integrable; indeed, it is not even conservative. Equations of this type possess attractors which describe the asymptotically stable structures. These no longer interact in particle-like manner, and are called solitary waves. It should be mentioned that even solitons may be shown to be stable if small perturbations to their shape relax back to the original shape by radiating energy to infinity [143]. The analysis is delicate and requires coming to grips with eigenvalues embedded in a continuous spectrum, once again a consequence of an unbounded domain.

The relation between the behaviour of integrable systems and the dynamics of the corresponding weakly damped driven system constitutes a major challenge to research. While it is true that the dissipation and forcing 'select' a subset of the solutions of the integrable system, much more can (and does) happen, particularly in the regime in which the dissipation, forcing and nonlinearity are all comparable. A particularly simple example of what may happen is provided by recent work of Higuera et al [132] on the damped driven nonlocal Schrödinger equation. In this work several global bifurcations are identified that are responsible for the presence of complex dynamics. Different global bifurcations are responsible for multi-pulse homoclinic orbits when the damping is small relative to the other two terms, as discussed in detail by Haller and Wiggins [131, 129, 130].

The final topic under the heading of \textit{weakly nonlinear theory} that I wish to mention is the issue of quasi-patterns. Such patterns are well known in physics, and have been generated in pattern-forming systems such as the Faraday system [74]. Of course a true quasi-pattern has structure over the whole plane, but the Fourier spectra of the observed patterns resemble those of true quasi-patterns. Such patterns are \textit{not} periodic in space, and in general form as a result of the presence of two or more incommensurate wave-numbers or wave-vectors. Because these wave-vectors are incommensurate there are strictly speaking no spatial resonances. As a result formal
asymptotic theory reveals no resonant terms and the amplitude equations that result contain to all orders only terms generated by self-interaction. The stability properties of the states that result are degenerate, since they possess zero eigenvalues that are not forced by symmetry. Most likely what is happening is the following: the near resonances lead to large values of the coefficients of (some of) the higher order terms, and this fact reduces the range of validity of the amplitude equation: the higher the order of the equation the smaller the region of validity. To my knowledge even the simplest problem on the real line, the interaction between wave-number 1 and an irrational wave-number $q$ is unsolved. What is necessary is to approximate $q$ via its continued fraction expansion approximations, $r_n/s_n$, and examine the radius of convergence of the resulting amplitude equations as $s_m \rightarrow \infty$. One can think of this type of theory as an extension of KAM theory (see, e.g., [30] and references therein), well studied in the context of temporal resonance, into the spatial domain.

**Fully nonlinear theory**

- Dynamics of fronts and defects, and their interaction
- Existence and stability of spatially localized structures
- Classification of instabilities of nonlinear structures
- Wavelength selection close to and far from onset
- Relation between discrete and continuum systems
- Relation between unbounded and large but bounded systems
- Frequency selection in open systems
- Transition to spatio-temporal chaos close to and far from onset

In the context of fully nonlinear theory there is a number of interesting and important topics, many of which also arise in the context of the amplitude or envelope equations derived within weakly nonlinear theory, now considered as field theories in their own right. These concern the basic mechanism for the selection of the speed of a front (these may be linear [59] or nonlinear [63]), and existence and interaction of defects, such as spiral waves in reaction-diffusion systems. Many excitable systems exhibit spirals that are initiated by a finite amplitude perturbation. How such spiral structures are related to bifurcation from the trivial state remains unknown. The existence of spirals in a plane was proved by Scheel [106] but such theorems do not provide much information about the structure of the spiral (frequency, wave-number and amplitude). It is of interest that observed spirals do not resemble the eigenfunctions associated to a Hopf bifurcation in planar systems described by polar coordinates. These either decay as $r^{-1/2}$ if the wave-number is real, or grow (or decay) exponentially if it is complex. Golubitsky et al [127] suggest that the latter case in fact provides a description of the core of the spiral, and that the
constant amplitude wave seen outside the core is a consequence of nonlinearities that affect the solutions as \( r \to \infty \) arbitrarily close to onset. In their view the core-spiral interface is a nonlinear front whose location is fixed by the imaginary part of the wave-number selected by the Hopf frequency. This wave-number is complex because in polar coordinates waves propagate preferentially in one direction (either outwards or inwards), so that the wave-number is selected by the requirement that the instability be \textbf{absolute} [155]. This approach also appears to explain the breakup of spiral waves into ‘chemical turbulence’ far from the core [155], as observed in some experiments [98]. These ideas await confirmation through quantitative studies of the relation between the spiral frequency and the core profile. It should be noted that spirals can also undergo a core instability [29].

Localized structures come in a variety of forms. Perhaps the most interesting are the \textit{oscillons} observed in granular media vibrated up and down [157]. These localized oscillations leave the rest of the system undisturbed, and may form ‘bound’ states that have been called dimers. These localized structures are believed to form as a consequence of a subcritical instability, essentially by the mechanism identified by Thual and Fauve [113]. However, there is no convincing explanation for their presence, largely because of an almost complete lack of understanding of the equations of motion governing granular media. There are many other systems where a subcritical bifurcation is responsible for the formation of steady localized states, including the recent discovery of ‘convections’, that is, localized regions of field-free convection in a fluid layer with a uniform vertical magnetic field imposed across it [5], and related structures in reaction-diffusion systems [141]. These states can all be thought of as homoclinic connections from the trivial state back to itself. For the stability of the resulting state it is essential that the state at \( x \to \pm \infty \) (i.e., the trivial state) be stable. It follows that such localized states can only be present for \( \mu < 0 \), where \( \mu = 0 \) denotes the primary bifurcation threshold. The situation is more interesting in systems undergoing a Hopf bifurcation since here homoclinic connections to the origin may be stable even for \( \mu > 0 \), cf. [110, 146]. Homoclinic connections can also describe localized wave-number and amplitude changes that may propagate, without change of profile, through a travelling wavefront, as in the examples computed by Spina et al [152]. Recent theory, based on the coupled complex Ginzburg-Landau equations (see Sect. 1.4) indicates that such ‘homoclinic holes’ should be unstable [117], although this is clearly not always the case. Evidently the predictions of the theory, for example concerning the speed of propagation of such structures, should be checked against both numerical experiments using model equations and other field equations, and against actual experiments. It is likely that there are many such solitary wave solutions, although most are presumably unstable [117, 105], so that mechanisms that stabilize these structures need
to be identified. Evans function techniques are invaluable for studies of the stability properties of these states [24, 25].

The question of stability of finite amplitude structures, be they periodic or localized, and their bifurcation is a major topic that requires new insights. The theory for the former requires developments of a theory describing bifurcation from group orbits of solutions while the latter requires a theory for the bifurcation that results when an unstable eigenvalue emerges from the continuum. Bifurcations from a group orbit often produce drifts along the group orbit [26, 31, 55], resulting in unexpected dynamics arising from steady state bifurcations [26]. The best known example of this bifurcation is the so-called parity breaking bifurcation. This is a bifurcation from a circle of reflection-symmetric equilibria. At the bifurcation (at \( \mu = 0 \), say) there is a zero eigenvalue whose eigenvector breaks the reflection symmetry of the equilibrium. This zero eigenvalue is in addition to the zero eigenvalue due to rotation invariance, and together they are responsible for the ensuing drift of the solution along the group orbit, either clockwise or counterclockwise; the drift speed vanishes as \( \mu^{1/2} \) as \( \mu \downarrow 0 \). Figure 1.1 shows some of the more complicated patterns that can be created in secondary bifurcations from a group orbit, this time from a pattern of standing hexagonal oscillations.

Bifurcations involving the continuum are also nonstandard but in a different way, and lead to a saturation amplitude that scales like \( \mu^2 \) instead of the more usual \( \mu^{1/2} \), where \( \mu > 0 \) is now the growth rate of the instability. This is the case, for example, in the beam-plasma instability [56]. Related phenomena, involving critical layer behaviour, arise in bifurcations in ideal shear flow as the shear flow profile is deformed. In particular instability of a vorticity defect in plane Couette flow is described by a nonlocal evolution equation closely related to the Vlasov equation [23], with related nonlocal equations, obtained by matching across a critical layer, describing the evolution of long wavelength perturbations of marginally stable shear flows [22]. Such flows are prepared by deforming the shear profile until the necessary and sufficient conditions for instability are satisfied, in the same way that a beam-plasma instability is triggered by a bump on the tail of the particle distribution in plasma physics. Similar phenomena also arise in the theory of phase-coupled oscillators [2]. Much remains to be learnt about these systems.

A major unsolved problem that has attracted attention at least since the 1960’s is the problem of wavelength selection in extended or unbounded systems. Linear theory identifies the wavelength of the first unstable disturbance. However, in the nonlinear regime the observed wavelength usually differs. Analysis, initiated by Busse [53], revealed that above the neutral stability curve \( \mu = \mu(k) \) there is usually an interval of stable wave-numbers \( k \), limited by various secondary instabilities, such as the Eckhaus, skewed varicose, oscillatory, and zigzag instabilities. In most cases these instabil-
Figure 1.1. Experimental and reconstructed surface wave patterns in silicone oil with two-frequency temporal forcing \((f_1 : f_2 = 2 : 3)\) arising from secondary instabilities of a small scale hexagonal pattern. (a) Hexagonal pattern on two scales. (b) Pattern with instantaneous triangular symmetry. (c,d) Theoretical reconstruction of (a,b). Courtesy A. Rucklidge.

Patterns leave a region of stable wave-numbers in the \((\mu,k)\) plane, nowadays called the Busse balloon, but do not select a unique wave-number. Yet
experiments usually follow a unique path through this region, as discussed in the context of Rayleigh-Bénard convection by Koschmieder [81]. Thus although this problem has led to profound developments such as the introduction of phase equations (e.g., the Kuramoto-Sivashinsky equation) into the theory of pattern formation, and the notion of sideband (Eckhaus, Benjamin-Feir) instabilities, the original problem remains unsolved. In fact, the phase description is \textit{the} appropriate description of \textit{finite} amplitude patterns [140, 142]. This approach goes beyond envelope equations, and has served to highlight the role played by instabilities in wavelength selection. Unfortunately, the description breaks down at defects, where the phase of the pattern is, by definition, undefined.

In many cases one supposes that a discretized version of a completely integrable PDE will behave the same way as the PDE. However, the discrete system may not be integrable, and one needs to understand in detail how the integrals of motion appear as the continuum limit is approached, and indeed how the dynamics of the discrete system approach those of the integrable system. These issues have a profound significance for accurate numerical simulation of integrable PDEs.

One is often tempted to suppose that distant boundaries have negligible effect on the process of pattern selection. While this may be so for steady state pattern-forming instabilities for which the boundaries shift the threshold for primary instability by $O(L^{-2})$ where $L$ measures the domain size [3], and modify the pattern substantially only near the boundary where matching to the boundary conditions is effected, systems supporting propagating waves behave quite differently. In such systems the wave is always in contact with the boundary, and the boundaries may exert profound influence. This is especially so when the waves have a preferred direction of propagation, as already mentioned in the context of our discussion of spiral waves. Since such waves cannot be reflected (all reflected waves are evanescent) the downstream boundary acts like an absorbing boundary. To overcome this dissipation the threshold for instability is shifted to $\mu = \mu_f > 0$. It turns out that $\mu_f$ is related to the threshold $\mu_n$ for \textit{absolute} instability in the unbounded system: $\mu_f = \mu_n + O(L^{-2})$. For values of $\mu$ between the convective instability threshold in an unbounded system, $\mu = 0$, and $\mu_f$, a disturbance originating at the upstream boundary grows as it propagates towards the downstream boundary, and piles up against it, increasing its wave-number to such an extent that it ultimately decays. Thus in systems with broken reflection symmetry the presence of boundaries, however distant, changes the threshold for instability by an $O(1)$ amount! Of course an $O(L)$ transient is present before the presence of the downstream boundary manifests itself, a property of the system that can be traced to the non-normality of the linear stability problem [20, 21]. This system therefore demonstrates that the mere existence of boundaries may have a fundamentally important effect, and thus represents a situation in
which the boundaries cannot be treated perturbatively however far apart they may be. Additional consequences of boundaries in this system are discussed in Section 1.6.

In problems of this type the upstream boundary serves as a ‘pace-maker’: this boundary selects the frequency which then determines the downstream amplitude and wave-number from a nonlinear dispersion relation. If the amplitude at the upstream boundary is sufficiently small this frequency will be close to the linear theory frequency (and hence the frequency $\omega_n$ predicted by the global instability condition); in other cases the frequency solves a nonlinear eigenvalue problem and must be computed numerically [156]. As discussed in Section 6 much remains poorly understood about these systems, particularly in cases with phase slips at the front that separates the upstream and downstream parts of the solution. Once again, we may think of the upstream part as the core of a spiral, and the downstream part as the fully developed (visible) spiral.

The final topic on the list is the transition to complex spatio-temporal behaviour in extended systems. While a certain number of routes to temporal chaos in low-dimensional dynamical systems have been identified and analyzed [18, 19], the situation in spatially extended systems is much more complex. These systems are plagued by very long transients (in fluid systems one typically has to wait several horizontal diffusion times before transients die out), and there is even the possibility that the characteristic time on which the system relaxes may effectively diverge at some scale $L$, such that for scales $\ell > L$ the system ‘never’ finds a stable equilibrium even though one (or more) may be present. Loosely speaking one can think of an ‘energy landscape’ that has so many (steady) states, most of which are non-stable, that the system spends forever wandering in this ‘landscape’. Analogies with annealing problems come to mind. Schmiele and Eckhardt [107, 108] have explored some of the consequences of this picture in the context of shear flow instability.

### 1.2 Pattern selection on lattices

As an example of the type of results that may be obtained when pattern selection problems are posed on lattices, we describe briefly the results for the Hopf bifurcation on a square lattice [151]. We take an isotropic, spatially homogeneous system in a plane, with a trivial state $\Psi(x_1, x_2) = 0$, and suppose that this state loses stability to a symmetry-breaking Hopf bifurcation at $\mu = 0$. The linear stability theory predicts the associated Hopf frequency $\omega_c$, and the critical wave-number $k_c = |k|$, assumed to be nonzero. In the following we impose periodic boundary conditions in two orthogonal directions, hereafter $x_1, x_2$, with period $2\pi/k_c$. This assumption reduces the symmetry group of the problem from $E(2)$, the Euclidean group of rotations
and translations in two dimensions, to the group $D_4 \rtimes T^2$, the semi-direct product of the symmetry of a square and a two-torus of translations, and selects the four wave-vectors \( k_1 = k_c(1,0), k_2 = k_c(0,1), k_3 = k_c(-1,0), k_4 = k_c(0,-1) \) from the circle of marginally stable wave-vectors. We may therefore write the most general marginally stable eigenfunction in the form

\[
\Psi(x_1, x_2) = (v_1(t)e^{ik_c x_1} + v_2(t)e^{ik_c x_2} + w_1(t)e^{-ik_c x_1} + w_2(t)e^{-ik_c x_2})f(y),
\]

where \( y \equiv x_3 \) denotes any transverse variables (if present). In the following we assume that the linear stability problem takes the form \( \dot{z} = \mu(\lambda)z \), where \( z \equiv (v_1, v_2, w_1, w_2) \in \mathbb{C}^4 \), and \( \mu(0) = i\omega_c, \quad \text{Re}(\mu'(0)) > 0 \). Here \( \lambda \) is the bifurcation parameter. It follows that the quantities \( |v_1|, |w_1| \) and \( |v_2|, |w_2| \) represent amplitudes of left- and right-travelling waves in the \((x_1, x_2)\) directions, respectively.

The group $D_4$ is generated by counterclockwise rotations by $90^\circ$ (hereafter \( \rho_{\pi/2} \)) and reflections $x_1 \to -x_1$. These symmetries act on $z \in \mathbb{C}^4$ as follows:

\[
\rho_{\pi/2} : \quad (v_1, v_2, w_1, w_2) \to (w_2, v_1, v_2, w_1), \quad \rho_{\pi/2} \in D_4, \tag{1.2}
\]

\[
\kappa : \quad (v_1, v_2, w_1, w_2) \to (w_1, v_2, v_1, w_2), \quad \kappa \in D_4. \tag{1.3}
\]

In addition spatial translations \((x_1, x_2) \to (x_1 + \theta_1 / k_c, x_2 + \theta_2 / k_c)\) act by

\[
(\theta_1, \theta_2) : \quad (v_1, v_2, w_1, w_2) \to (e^{i\theta_1}v_1, e^{i\theta_2}v_2, e^{-i\theta_1}w_1, e^{-i\theta_2}w_2),
\quad (\theta_1, \theta_2) \in T^2. \tag{1.4}
\]

Finally, in normal form the dynamical equations will commute with an $S^1$ phase shift symmetry in time acting by

\[
\phi : \quad z \to e^{i\phi}z, \quad \phi \in S^1. \tag{1.5}
\]

As a result the full symmetry group of the dynamical equations near $\lambda = 0$ is $\Gamma \equiv D_4 \rtimes T^2 \times S^1$. An examination of the action of $\Gamma$ on $z \in \mathbb{C}^4$ shows that there are four axial isotropy subgroups of $\Gamma$ with two-dimensional fixed points subspaces. The equivariant Hopf theorem [77] guarantees the existence of primary branches of solutions with the symmetries of these subgroups. These solutions, listed in Table 1, are called standing squares (SS), and travelling (TR), standing (SR) and alternating rolls (AR). However, in the present example there is in open regions of parameter space a fifth primary solution branch whose existence is not revealed by the abstract theory. Such branches are sometimes called sub-maximal in the present case the sub-maximal branch corresponds to standing cross-rolls (SCR). At present the only way of locating sub-maximal branches is by explicit calculation.
For this purpose we write down the most general set of equations commuting with the above symmetries, truncated at third order [151]

\[
\begin{align*}
\dot{v}_1 &= \mu v_1 + (a|v_1|^2 + b|w_1|^2 + c|w_2|^2)v_1 + dv_2w_2\dot{v}_1 \\
\dot{v}_2 &= \mu w_2 + (a|v_2|^2 + b|w_2|^2 + c|v_1|^2)v_2 + dv_1w_1\dot{v}_2 \\
\dot{w}_1 &= \mu w_1 + (a|w_1|^2 + b|v_1|^2 + c|v_2|^2)w_1 + dw_2v_2\dot{w}_1 \\
\dot{w}_2 &= \mu v_2 + (a|w_2|^2 + b|v_2|^2 + c|v_1|^2)w_2 + dv_1w_1\dot{w}_2.
\end{align*}
\]

(1.6) (1.7) (1.8) (1.9)

The construction of these equations is algorithmic, and requires first the construction of the Hilbert basis of invariant functions and then the Hilbert basis of equivariant vector fields [151, 123]. These results can be used to generate the required amplitude equations to any desired order, via a procedure that can be automated. These equations can not only be used to compute the five nontrivial primary branches, but also to determine their stability properties with respect to perturbations on the chosen lattice, i.e., with respect to perturbations in \( (v_1, v_2, w_1, w_2) \in \mathbb{C}^4 \). At present there are no techniques for the a priori exclusion of sub-maximal primary branches, a fact that serves as an obstruction to a completely group-theoretic analysis of the primary bifurcation. In the present case the sub-maximal branch exists in an open region in coefficient space, and is always unstable [153, 151], although in other problems no restrictions on the existence and stability properties of the sub-maximal solutions are present [37].

The corresponding analysis for the steady bifurcation on a square lattice was first done by Swift [153], see also [115, 76], while those for the hexagonal lattice were obtained by Golubitsky et al [128]. The former case describes the competition between rolls and squares, and shows that at most one of these states can be stable near onset, and that the stable state is the one with the larger amplitude. On the hexagonal lattice the generic situation leads to a primary bifurcation with no (locally) stable branches. Hexagonal solutions bifurcate transcritically but are unstable on both sides of \( \mu = 0 \); rolls are either supercritical or subcritical but in either case are also unstable. This result is an example of a general result: primary branches in amplitude equations possessing quadratic equivariants are unstable whenever these equivariants do not vanish in the corresponding fixed point subspace. The hexagons on either side of \( \mu = 0 \) differ. In the convection context we speak of \( H^\pm \), with \( H^+ \) representing hexagons with rising fluid in the centre, while \( H^- \) denotes hexagons in which the fluid descends in the centre. Of these the subcritical one usually gains stability at a secondary saddle-node bifurcation. This, and secondary bifurcations to a branch of triangles can be studied by looking at systems with a weakly broken mid-plane reflection symmetry [128]. It is important that hexagons and triangles not be confused. Unfortunately, these solutions are distinguished unambiguously only if the basic instability wavelength is known.

The Hopf bifurcation on a line is a special case of the Hopf bifurca-
tion in a plane just discussed (set $v_2 = w_2 = 0$) and in amplitude-phase variables reduces to the steady bifurcation with $D_4$ symmetry [153], with travelling (rotating) waves taking the place of rolls, and standing waves taking the place of squares. Consequently here too at most one of these states can be stable near onset, and the stable state is the one with the larger amplitude. The corresponding results for the hexagonal lattice were worked out by Roberts et al [103], who computed the 11 primary branches guaranteed by the equivariant branching lemma together with their stability properties. Rotating lattices have also been analyzed. A large number of applications of these results have now been worked out, but a major unsolved problem remains: there is no rigorous theory that would allow us to establish the stability of squares with respect to hexagonal perturbations and vice versa, essentially because there is no spatially periodic lattice that accommodates both solution types. Only in systems described by a potential can one compare the extrema of the potential corresponding to these states, and hence determine which of these states is stable (or metastable if both correspond to minima). It should be noted that certain sufficiently low order truncations of the amplitude equations for steady state bifurcations do yield gradient vector fields. This is, however, an artifact of the truncation, and amplitude equations do not in general possess variational structure.

A possible approach to the vexed problem of the competition between squares and hexagons is based on higher-dimensional representations of the symmetry groups of planar lattices. In the case of the square lattice such a representation arises when periodic boundary conditions with a larger period are employed. For example, the reciprocal square lattice can intersect the circle of marginally stable wave-vectors in 8 instead of 4 places. These wave-vectors are parametrized by two relatively prime integers $\alpha > 0$, $\beta > 0$, with the critical wave-vectors given by $K_1 = \alpha k_1 + \beta k_2$, $K_2 = -\beta k_1 + \alpha k_2$, $K_3 = \beta k_1 + \alpha k_2$, $K_4 = -\alpha k_1 + \beta k_2$, where $k_1 = (1, 0)k_c$, $k_2 = (0, 1)k_c$, and produce a countably infinite number of eight-dimensional representations of $D_4 \oplus T^2$. Group-theoretic results show that in this case there are 6 axial isotropy subgroups [65]; the new patterns that are possible include super-squares, anti-squares and two types of rectangular patterns. A similar theory for the 12-dimensional representation of $D_4 \oplus T^2$ is available, and the stability properties of all the primary branches on both lattices have been worked out [67]. Super-triangles have been observed in the Faraday system [92, 36], indicating that these higher-dimensional representations are indeed relevant to the problem of pattern formation. Stable super-squares and anti-squares can also be located in systems of reaction-diffusion equations [46]. The abstract stability results enable one to compute the stability of hexagons with respect to rectangular patterns that are almost square. However, the relevance of these results to the ultimate question of relative stability between hexagons and squares.
remains unclear. Related results for the Hopf bifurcation on the square super-lattice are given by Dawes [62].

Certain of the possible three-dimensional lattices have also been considered. The steady state bifurcations on the simple cubic, face-centred cubic and body-centred cubic lattices were analyzed by Callahan and Knobloch [123]. At \( \mu = 0 \) these bifurcations have a zero eigenvalue of multiplicity 6, 8 and 12, respectively, resulting from the selection of 6, 8 or 12 wave-vectors from the sphere of marginally stable wave-vectors. Except for the simple cubic case (which corresponds to a wreath product group) the remaining two cases are plagued by sub-maximal branches which greatly complicate the analysis of these bifurcations. The corresponding calculations for the Hopf bifurcation on the simple cubic lattice were performed by Dias [64] exploiting its wreath product structure, and for the face-centred cubic lattice by Callahan [122]. None of the higher-dimensional representations of the cubic lattices have been considered, and neither have the remaining lattices except to identify the axial isotropy subgroups [65], and hence the primary solutions guaranteed by the equivariant branching lemma [66].

1.3 Imperfection sensitivity

The results of equivariant bifurcation theory depend of course on the presence of the assumed symmetry. This symmetry may represent the result of an idealization of a physical system, or it may be the consequence of imposed periodic boundary conditions as in the examples described above. In either case a natural question arises concerning the robustness of the results when the assumed symmetry is broken. Forced symmetry breaking is almost inevitable in physical situations, and while the dynamics of highly symmetric systems may be of interest in their own right, applications demand that one attempts to identify those aspects of equivariant dynamics that persist under (small) perturbations of the assumed symmetry. This point of view is not only relevant to pattern formation problems, where boundary conditions may destroy the assumed spatial periodicity, or a small feed gradient may destroy the assumed homogeneity (and isotropy) of the system, but also in structural engineering, where attempts to build redundancy into a system require understanding the consequences of the loss of a particular strut or support beam [79]. At present there is no general theory that allows one to identify the consequences of such symmetry-breaking imperfections. The basic idea is simple: one seeks to embed the equivariant dynamical system in a larger class of vector fields with smaller symmetry. For the pitchfork bifurcation this notion leads to the universal unfolding of the pitchfork [77], and shows that the effects of all possible imperfections that break reflection symmetry are captured by just two unfolding parameters. A general formulation of problems of
this type is given by Lauterbach and Roberts [13]; Callahan and Knobloch [124] introduce the notion of co-equivariance and use it to generate the most general isotropy-breaking contributions to the equations describing a steady state bifurcation on the hexagonal lattice. Certain other examples have been worked out rigorously. Of these the best known is the behaviour of a steady state symmetry-breaking bifurcation with D$_4$ symmetry under perturbations that reduce the symmetry to D$_2$. This situation arises naturally in pattern selection problems in containers of square and nearly square (rectangular) cross-section [51], and in the normal form describing a symmetry-breaking Hopf bifurcation with O(2) symmetry when the O(2) symmetry is broken down to SO(2) [115, 57]. A related but distinct example of this type of problem arises in systems with Neumann boundary conditions on a square domain. Such problems can be embedded in the corresponding problem on the square lattice, an embedding that is destroyed when the shape of the domain becomes non-square even though it may retain D$_4$ symmetry. The resulting breakup of the primary bifurcation is discussed by Crawford [43] and confirmed experimentally by Crawford et al. [44].

In the absence of a general theory a reasonable approach appears to be to include only the dominant symmetry-breaking terms. While this approach may not capture all possible effects of the loss of symmetry it seems plausible that it provides a good guide to the robustness of the results with the full symmetry. This approach is especially enlightening when it comes to problems with continuous symmetries. Continuous symmetries allow simple types of dynamics, such as rotating waves, because the phase of the wave decouples from the equation for its amplitude. When the translation invariance is broken (by spatial inhomogeneity or the presence of boundaries) the spatial phase couples to the amplitude, thereby raising the order of the dynamical system. This coupling in turn is often responsible for the introduction of global bifurcations into the dynamics, and these are often associated with the presence of chaotic dynamics. Thus symmetry-breaking imperfections can produce complex dynamics in systems that would otherwise behave in a simple fashion. Several examples of this type of behaviour have been worked out, for the Hopf bifurcation with O(2) symmetry broken down to Z$_2$ [60, 133], and for the Hopf bifurcation with D$_4$ symmetry broken down to D$_2$ [138, 139]. We describe here the latter case. This example is believed to be relevant to experiments on convection in $^3$He/$^4$He mixtures in a finite but extended container in which the convective heat transport immediately above threshold ($\tau^2 \equiv (R - R_c)/R_c = 3 \times 10^{-4}$) may take place in a sequence of irregular bursts of large dynamic range despite constant heat input [112].

Numerical simulations of the two-dimensional equations in a container of aspect ratio $L = 16$ suggest that these bursts involve the interaction between the first two modes of the system [80, 121]. These have opposite
parity, and because the neutral stability curve for the unbounded system has a parabolic minimum, set in in close succession as the bifurcation parameter is increased. Near threshold the perturbation from the trivial state then takes the form

$$\Psi(x, y, t) = \epsilon \text{Re} \left\lbrace z_+ f_+(x, y) + z_- f_-(x, y) \right\rbrace + O(\epsilon^2),$$

(1.10)

where $\epsilon \ll 1$, $f_{\pm}(-x, y) = \pm f_{\pm}(x, y)$, and $y$ again denotes transverse variables. The complex amplitudes $z_{\pm}(t)$ then satisfy the normal form equations [93]

$$\dot{z}_\pm = [\lambda \pm \Delta \lambda + i(\omega \pm \Delta \omega)]z_{\pm} + A(|z_+|^2 + |z_-|^2)z_{\pm} + B|z_{\pm}|^2 z_{\pm} + C z_{\pm} z_{\pm}^2. \quad (1.11)$$

In these equations the nonlinear terms have identical (complex) coefficients because of an approximate interchange symmetry between the odd and even modes when $L \gg 1$. When $\Delta \lambda = \Delta \omega = 0$ the resulting equations coincide with equations (1.6)-(1.9) in the standing wave subspace $v_1 = u_1$, $v_2 = u_2$, and have $D_4$ symmetry. This symmetry is weakly broken whenever $\Delta \lambda \neq 0$ and/or $\Delta \omega \neq 0$, a consequence of the finite aspect ratio of the system [93].

To identify the bursts we introduce the change of variables

$$z_{\pm} = \rho^{-1/4} \sin \left( \frac{\theta}{2} + \frac{\pi}{4} \pm \frac{\pi}{4} \right) e^{i(\pm \phi + \psi)/2}$$

and a new time-like variable $\tau$ defined by $d\tau/dt = \rho^{-1}$. In terms of these variables equations (1.11) become

$$\frac{d\rho}{d\tau} = -\rho[2A_R + B_R(1 + \cos^2 \theta) + C_R \sin^2 \theta \cos 2\phi] - 2(\lambda + \Delta \lambda \cos \theta) \rho^2$$

(1.12)

$$\frac{d\theta}{d\tau} = \sin \theta[\cos \theta(-B_R + C_R \cos 2\phi) - C_I \sin 2\phi] - 2\Delta \lambda \rho \sin \theta$$

(1.13)

$$\frac{d\phi}{d\tau} = \cos \theta(B_I - C_I \cos 2\phi) - C_R \sin 2\phi + 2\Delta \omega \rho,$$

(1.14)

where $A = A_R + iA_I$, etc. There is also a decoupled equation for $\psi(t)$ so that fixed points and periodic solutions of equations (1.12)-(1.14) correspond, respectively, to periodic solutions and two-tori in equations (1.11). In the following we measure the amplitude of the disturbance by $r \equiv |z_+|^2 + |z_-|^2 = \rho^{-1}$; thus $\rho = 0$ corresponds to infinite amplitude states. Equations (1.12)-(1.14) show that the restriction to the invariant subspace $\Sigma \equiv \{ \rho = 0 \}$ is equivalent to taking $\Delta \lambda = \Delta \omega = 0$ in (1.13)-(1.14). Since the resulting $D_4$-symmetric problem is a special case of equations (1.6)-(1.9) we may use Table 1 to conclude that there are three generic types of
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fixed points [154]: SS solutions with $\cos \theta = 0, \cos 2\phi = 1$; AR solutions with $\cos \theta = 0, \cos 2\phi = -1$; and SR solutions with $\sin \theta = 0$. In the following we refer to these solutions as $u$, $v$, $w$, respectively, to emphasize that their physical interpretation is now quite different. In fact, in the binary fluid context these solutions represent, respectively, mixed parity travelling wave states localized near one of the container walls, mixed parity chevron states, and pure even ($\theta = 0$) or odd ($\theta = \pi$) parity chevron states. The chevron states consist of waves propagating outwards from the centre of the container (or inwards from the boundaries) that are either in phase at the boundaries (even parity) or out of phase (odd parity). In a finite container these are the only states that bifurcate from the trivial (conduction) state. Depending on $A$, $B$ and $C$ the subspace $\Sigma$ may contain additional submaximal fixed points (SCR) as well as limit cycles [154]. In our scenario, a burst occurs for $\lambda > 0$ when a trajectory follows the stable manifold of a fixed point (or a limit cycle) $P_1 \in \Sigma$ that is unstable within $\Sigma$. The instability within $\Sigma$ then kicks the trajectory towards another fixed point (or limit cycle) $P_2 \in \Sigma$. If this point has an unstable $\rho$ eigenvalue the trajectory escapes from $\Sigma$ towards a finite amplitude ($\rho > 0$) state, forming a burst. If $\Delta \lambda$ and/or $\Delta \omega \neq 0$ this state may itself be unstable to perturbations of type $P_1$ and the process then repeats. This bursting behaviour is thus associated with a codimension one heteroclinic cycle between the infinite amplitude solutions $P_1$ and $P_2$ [139].

For the heteroclinic cycle to form it is necessary that at least one of the branches in the $D_4$-symmetric system be subcritical ($P_1$) and one be supercritical ($P_2$). For the parameters of figure 1.2 the $u$ solutions are subcritical while $v$, $w$ are supercritical when $\Delta \lambda = \Delta \omega = 0$ [138] and two of the resulting cycles are shown in figure 1.2. In each case the trajectory reaches infinity in finite time and the heteroclinic cycle therefore represents infinite amplitude bursts of finite duration [139]. Consequently the time-averaged amplitude $\langle r \rangle$ may be dominated by the time spent near finite amplitude states, as in figure 1.3(a). Although neither of these solutions is in fact stable, they (and others like them) are responsible for the wealth of burst-like behaviour exhibited by this system. Figure 1.3 is an attempt to summarize some of this complexity in the form of a bifurcation diagram, but only the low period solutions have been followed. Much of it can be traced to the Shil’nikov-like properties of the dominant heteroclinic cycles [139].

The solutions shown in figure 1.2 are both (nearly) infinite period librations, characterized by bounded $\phi$. But periodic trajectories in the form of rotations ($\phi$ increasing without bound) are also possible, and (1.4) shows an example of chaotic bursts generated by a stable chaotic rotation. Figure 1.5 shows the physical manifestation of the bursts arising from rotations and librations in the form of space-time plots using the approximate
Fig. 1.2. Numerically obtained approximate heteroclinic cycles for $\Delta \lambda = 0.03$, $\Delta \omega = 0.02$, $A = 1-1.5i$, $B = -2.8+5i$, and (a) $C = 1+i$, (b) $C = 0.9 + i$ present at (a) $\lambda = 0.0974$ and (b) $\lambda = 0.08461$. The + signs indicate infinite amplitude $u$ states responsible for the bursts, while the squares indicate infinite amplitude $v$ states and the diamonds finite amplitude states.

eigenfunctions

$$f_{\pm}(x) = \{e^{-\gamma x + ix} \pm e^{\gamma x - ix}\} \cos \frac{\pi x}{L},$$

where $\gamma = 0.15 + 0.025i$, $L = 80$ and $-\frac{L}{2} \leq x \leq \frac{L}{2}$. The bursts in figure 1.5(a) are generated as a result of successive visits to different but symmetry-related infinite amplitude $u$ solutions; in figure 1.5(b) the generating trajectory makes repeated visits to the same infinite amplitude $u$ solution. The former state is typical of the blinking state identified in binary fluid and doubly diffusive convection in rectangular containers [90, 101]. It is likely that the irregular bursts reported in [112] are due to such a state. The latter, a twinkling state, may be stable but often coexists with stable chevron-like states which are more likely to be observed in experiments in which the Rayleigh number is ramped upwards, cf. figure 1.3(a).

The bursts described above are the result of oscillations in amplitude between two modes of opposite parity and “frozen” spatial structure. Consequently the above burst mechanism applies in systems in which bursts occur very close to threshold. This occurs not only in the convection experiments already mentioned but also in the mathematically identical Taylor-Couette system where counter-propagating spiral vortices play the same role as travelling waves in convection [47, 48]; see also Mullin, this volume.
Figure 1.3. Partial bifurcation diagrams for (a) $C = 1 + i$ and (b) $C = 0.9 + i$ with the remaining parameters as in figure 1.2 showing the time-average of $r$ for different solutions as a function of $\lambda$. Solid (dashed) lines indicate stable (unstable) solutions. The branches labelled $u$, $v$, $w$, and $qp$ (quasiperiodic) may be identified in the limit of large $|\lambda|$ with branches in the corresponding diagrams when $\Delta \lambda = \Delta \omega = 0$ (insets). All other branches correspond to bursting solutions which may be blinking or winking states (figure 1.5). Circles, squares, and diamonds in the diagram indicate Hopf, period-doubling, and saddle-node bifurcations, respectively.

In slender systems, such as the convection system described above or a long Taylor-Couette apparatus, a large aspect ratio $L$ is required for the presence of the approximate $D_4$ symmetry. If the size of the $D_4$ symmetry-breaking terms $\Delta \lambda$, $\Delta \omega$ is increased too much the bursts fade away and are replaced by smaller amplitude, higher frequency states [139]. Indeed,
if $\Delta \omega \gg \Delta \lambda$ averaging eliminates the $C$ terms responsible for the bursts. From these considerations, we conclude that bursts will not be present if $L$ is too small or $\epsilon$ too large. It is possible that the burst amplitude can become large enough that secondary instabilities not captured by the Ansatz (1.10) can be triggered. Such instabilities could occur on very different scales and result in turbulent rather than just large amplitude bursts. It should be emphasized that the physical amplitude of the bursts is $O(\epsilon)$ and so approaches zero as $\epsilon \downarrow 0$, cf. (1.10). Thus despite their large dynamical range (i.e., the range of amplitudes during the bursts) the bursts are fully and correctly described by the asymptotic expansion that leads to equations (1.11). In particular, the mechanism is robust with respect to the addition of small fifth order terms [139].

### 1.4 Coupled Ginzburg-Landau equations

We consider a translation-invariant system on the real line with periodic boundary conditions, undergoing a symmetry-breaking Hopf bifurcation from the trivial state, i.e., $k_c \neq 0$. Traditionally, one writes

$$ u(x,t) = \epsilon u(y) [A(X,T)e^{i(\omega t + k_c x)} + B(X,T)e^{i(\omega t - k_c x)} + \text{c.c.}] + \ldots \quad (1.15) $$

where $\epsilon \ll 1$, $X = \epsilon x$, $T = \epsilon^2 t$ denote slow variables, and $y$ denotes any transverse variables. We can think of the parameter $\epsilon$ as specifying the distance to the threshold of the primary instability, e.g., $(R - R_c)/R_c = \epsilon^2 \mu$ as in Section 1.3. The asymptotic theory applies on scales $L$ such that $X = O(1)$, i.e., $L = O(\epsilon^{-1})$. It follows therefore that the resulting theory applies whenever $(R - R_c)/R_c = O(L^{-2}) \ll 1$. With the above Ansatz substituted into the field equations, one recovers at leading order the linear stability problem for $u = 0$, and hence determines $k_c$ and $\omega_c$, as well as $R_c$. 

**Figure 1.4.** Time series and peak-to-peak plot showing bursts from chaotic rotations at $\lambda = 0.072$ for the parameters of figure 1.2.
Figure 1.5. The perturbation $\Psi$ from the trivial state represented in a space-time plot showing (a) a periodic blinking state (in which successive bursts occur at opposite sides of the container) corresponding to a stable rotation at $\lambda = 0.1$, and (b) the periodic winking state (in which successive bursts occur at the same side of the container) corresponding to a stable libration at $\lambda = 0.1253$. 
From the remaining terms one obtains

\[ A_T + \frac{c_2}{\epsilon} A_X = (\mu + i\omega') A + (a|A|^2 + b|B|^2) A + cA_{XX} + O(\epsilon), \quad (1.16) \]

\[ B_T - \frac{c_3}{\epsilon} B_X = (\mu + i\omega') B + (a|B|^2 + b|A|^2) B + cB_{XX} + O(\epsilon), \quad (1.17) \]

where \( c_2 = d\omega/dk \) is the group velocity at \( k = k_c \), \( \omega' = \omega(\mu) - \omega \), and \( a \), \( b \) and \( c \) are (complex) constants. These equations are often used as models of systems undergoing a symmetry-breaking Hopf bifurcation. However, when \( c_2 = O(1) \) (the usual case) they are clearly inconsistent as \( \epsilon \to 0 \). For this reason one typically has to assume that the system is near a codimension two point such that \( c_2 = O(\epsilon) \). Only in this case do the above equations make asymptotic sense. But there is no reason why one needs to restrict attention to this case. More generally, if \( c_2 = O(1) \) one sees that the advection term dominates the nonlinear terms, indicating that the timescale for advection at the group velocity is much faster than the timescale \( T \) on which the system equilibrates. We therefore introduce the intermediate timescale \( \tau = \epsilon t \), and write \( A = A(X, \tau, T; \epsilon) \equiv A_0 + \epsilon A_1 + \ldots \), \( B = B(X, \tau, T; \epsilon) \equiv B_0 + \epsilon B_1 + \ldots \). At \( O(\epsilon^3) \) one now finds

\[ A_{0\tau} + c_2 A_{0X} = 0, \quad B_{0\tau} - c_2 B_{0X} = 0, \quad (1.18) \]

indicating that \( A_0 \equiv A_0(\xi, T) \), \( B_0 \equiv B_0(\eta, T) \), where \( \xi \equiv X - c_2 \tau \), \( \eta \equiv X + c_2 \tau \), while at \( O(\epsilon^2) \) one obtains

\[ A_{1\tau} + c_2 A_{1X} = -A_{0T} + (\mu + i\omega') A_0 + (a|A_0|^2 + b|B_0|^2) A_0 + cA_{0\xi} \quad (1.19) \]

\[ B_{1\tau} - c_2 B_{1X} = -B_{0T} + (\mu + i\omega') B_0 + (a|B_0|^2 + b|A_0|^2) B_0 + cB_{0\eta}. \quad (1.20) \]

The solvability condition for \( A_1 \), guaranteeing that \( \epsilon A_1 \) remains small relative to \( A_0 \) for \( T = O(1) \), yields the required evolution equation for \( A_0 \):

\[ A_{0T} = (\mu + i\omega') A_0 + (a|A_0|^2 + b|B_0|^2) A_0 + cA_{0\xi}, \quad (1.21) \]

and similarly

\[ B_{0T} = (\mu + i\omega') B_0 + (a|B_0|^2 + b|A_0|^2) B_0 + cB_{0\eta}. \quad (1.22) \]

Here

\[ \langle |A_0|^2 \rangle^\xi = \frac{1}{P} \int_0^P |A_0|^2 \, d\xi, \quad \langle |B_0|^2 \rangle^\eta = \frac{1}{Q} \int_0^Q |B_0|^2 \, d\eta, \quad (1.23) \]

where \( P \) and \( Q \) are the periods in \( \xi \) and \( \eta \) (perhaps infinite). Thus it is the presence of the intermediate advection timescale that is responsible for the nonlocal terms \( \langle \ldots \rangle \) in the envelope equations (1.21)-(1.22). These equations, derived by Knobloch and De Luca [133] for dissipative systems,
and by Chikwendu and Kevorkian [125], and Knobloch and Gibbon [136] for conservative systems, have been shown to be correct, in the sense that their solutions track those of the original field equations for \( T = O(1) \) on spatial scales \( X = O(1) \) [145]. Detailed stability results for these equations are given by Knobloch [134]. More recently, Riecke and Kramer [149] have explored the relation between equations (1.16)-(1.17) and the asymptotic equations (1.21)-(1.22), and showed that the solutions of the latter are indeed correct as a limit of the former, but that they may only apply in a small neighbourhood of \( R = R_c \). Whether the local equations (1.16)-(1.17) have any validity beyond qualitative outside of this neighbourhood remains at present a matter of conjecture. Pressure effects in incompressible flows [11] and in chemical kinetics [28] also yield nonlinear equations of the above type, as does mass conservation [144]. Of course, nonlocal equations are harder to work with, but it should be clear that they admit a larger class of solutions than local equations, and are of interest therefore for this reason alone, Duan et al [73] have proved the existence of a finite-dimensional inertial manifold for the single nonlocal complex Ginzburg-Landau equation

\[
CT = \mu C + (a|C|^2 + b|C|^2)C + c|C|^2, \quad (1.24)
\]

that describes the dynamics of the system (1.21)-(1.22) in the invariant subspace \( A_0 = B_0 \equiv C \). Such equations also arise in the narrow gap Taylor-Couette system [11], as well as describing ferromagnetic instabilities [12]. However, as already mentioned, existing derivations of these equations all ignore the presence of boundaries, and additional work must be carried out to derive the correct boundary conditions appropriate to specific applications. For the Hopf bifurcation in one spatial dimension, two boundary conditions must be imposed at each boundary [14]. These boundaries do more than simply break translation invariance, since waves can also be reflected from them suffering a phase shift, in addition to losing energy.

It is important to remark that there is another way of rationalizing the expansion (1.16)-(1.17). This is to choose instead \((R - R_c)/R_c = \epsilon \mu = O(L^{-1})\), where \( L \) again measures the domain size. In this case the wave amplifies and interacts with the boundaries on the timescale \( \tau \) and reaches a larger amplitude \( u = O(\epsilon^{1/2}) \). One now obtains a pair of hyperbolic equations of the form

\[
A_\tau + c_\delta A_X = (\mu + i \omega') A + (a|A|^2 + b|B|^2) A + O(\epsilon), \quad (1.25)
\]

\[
B_\tau - c_\delta B_X = (\mu + i \omega') B + (a|B|^2 + b|A|^2) B + O(\epsilon). \quad (1.26)
\]

This time only two boundary conditions are required, and these are given by

\[
A(0, t) = r B(0, t), \quad B(L, t) = r A(L, t). \quad (1.27)
\]
Here \( r \) is a (calculable) complex reflection coefficient [14]. The resulting equations can be written as pair of coupled equations for the real amplitudes \( |A|, |B| \); their solutions describe not only waves that propagate from one end of container to the other, but also waves that "blink", i.e., bounce back and forth between them [15]. Waves of this type have been seen in numerical simulations of doubly diffusive convection [9], and in experiments on both the oscillatory instability of convection rolls [10] and in binary fluid convection [90]. In the latter the subcritical nature of the instability complicates the dynamics substantially [121].

The above equations represent the effects of a global nonlocality: the global coupling does not fall off with distance. Yet in many cases (for example, in fluids that are not strictly incompressible) we would expect the finite speed of propagation of pressure waves to introduce a kernel with a cut-off into equations of this type. Equations with this weaker type of global coupling arise naturally in coupled oscillator systems as discussed by Professor Kuramoto elsewhere in this volume. The introduction of the new length scale, the effective range of the global coupling, has interesting and nontrivial consequences for the dynamics of these systems. A related equation arises as a singular limit of a pair of coupled reaction-diffusion equation with disparate timescales [6].

### 1.5 The nearly-inviscid Faraday system

The nearly inviscid Faraday system provides another example of a system described by nonlocal amplitude equations. In this system a container of liquid is vibrated vertically; when the amplitude of the acceleration of the container exceeds a threshold value, the flat surface breaks up into a pattern of standing waves [89, 75, 91]. In the absence of forcing such surface gravity-capillary waves decay of an \( O(C^{-1/2}) \) timescale, where 
\[
C = \nu /(gh^3 + Th/\rho)^{1/2} \ll 1, \]
while hydrodynamic (i.e., viscous) modes decay yet more slowly, on an \( O(C^{-1}) \) timescale [94]. Here \( g \) is the gravitational acceleration, \( T \) is the coefficient of surface tension, \( \rho \) is the density and \( \nu \) is the kinematic viscosity. Since the viscous modes decay so slowly they are easily excited by the vertical vibration; this excitation takes the form of a mean flow. In periodic domains in which the length of the domain is large relative to the wavelength of the instability this mean flow contains both viscous and inviscid contributions, and both couple to the amplitude equations for the surface waves. Traditionally such amplitude equations are derived using a velocity potential formulation. However, this formulation implicitly excludes large-scale streaming flows that may be driven by the time-averaged Reynolds stress in the oscillatory boundary layers at the container walls, or at the free surface. Recently a systematic asymptotic technique has been developed that includes such flows in a self-consistent
manner [119], and leads to a new class of (nonlocal) pattern-forming am-
plitude equations. These developments are summarized below.

We consider a container in the form of a right cylinder with horizontal
cross-section Σ filled level with the brim at z = 0. In this geometry the
contact line is pinned at the lateral boundary and complications associated
with contact line dynamics are reduced. The governing equations, non-
dimensionalized using the unperturbed depth h as unit of length and the
gravity-capillary time [g/h + T/(ρh^3)]^{-1/2} as unit of time, are

\[
\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla \Pi + C \Delta \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0,
\]

if \((x, y) \in \Sigma, \quad -1 < z < f,\)

\[
\mathbf{v} = 0 \text{ if } z = -1 \text{ or } (x, y) \in \partial \Sigma, \quad f = 0 \text{ if } (x, y) \in \partial \Sigma,
\]

\[
\mathbf{v} \cdot \mathbf{n} = (\partial f/\partial t)(e_z \cdot \mathbf{n}), \quad \left[(\nabla \mathbf{v} + \nabla \mathbf{v}^\top) \cdot \mathbf{n}\right] \times \mathbf{n} = 0 \quad \text{at } z = f,
\]

\[
\Pi - |\mathbf{v}|^2/2 - (1 - S)f + S \Delta \cdot [\nabla \mathbf{v}/(1 + |\nabla \mathbf{v}|^2)^{1/2}] = C[(\nabla \mathbf{v} + \nabla \mathbf{v}^\top) \cdot \mathbf{n}] \cdot \mathbf{n} - 4\mu \omega^2 f \cos 2\omega t \quad \text{at } z = f,
\]

where \(\mathbf{v}\) is the velocity, \(f\) is the associated vertical deflection of the free
surface (constrained by volume conservation), \(\Pi = p + |\mathbf{v}|^2/2 + (1 - S)z - 4\mu \omega^2 z \cos 2\omega t\) is the hydrostatic stagnation pressure, \(\mathbf{n}\) is the outward unit
normal to the free surface, \(e_z\) is the upward unit vector, and \(\partial \Sigma\) denotes the
boundary of the cross-section \(\Sigma\) (i.e., the lateral walls). The real parameters
\(\mu > 0\) and \(2\omega\) denote the amplitude and frequency of the forcing, and \(S = T/(T + \rho gh^2)\) is the gravity-capillary balance parameter. Thus \(0 \leq S \leq 1\)
with \(S = 0\) and \(S = 1\) corresponding to the purely gravitational limit
\((T = 0)\) and the purely capillary limit \((g = 0)\), respectively.

In the (nearly inviscid, nearly resonant, weakly nonlinear) regime

\[
C \ll 1, \quad |\omega - \Omega| \ll 1, \quad \mu \ll 1,
\]

(1.28)

where \(\Omega\) is an inviscid eigenfrequency of the linearized problem around the
flat state, the vorticity contamination of the bulk from the boundary layers
at the walls and the free surface remains negligible for times that are not
too long, and the flow in the bulk is correctly described by an inviscid
formulation but with boundary conditions determined by a boundary layer
analysis. In general this flow consists of an inviscid part and a viscous part.

In this regime it is possible to perform a multi-scale analysis of the
governing equations using \(C, L^{-1}\) and \(\mu\) as unrelated small parameters.
Here \(L\) is the dimensionless length of the container. The problem is simplest
in two dimensions where we can use a stream function formulation, i.e.,
we write \(\mathbf{v} = (-\psi_z, 0, \psi_x)\). We focus on two well-separated scales in both space
\((x \sim 1 \text{ and } x \gg 1)\) and time \((t \sim 1 \text{ and } t \gg 1)\), and derive equations for
small, slowly-varying amplitudes \(A\) and \(B\) of left- and right-propagating
waves defined by

\[
f = e^{i\omega t} (A e^{ikx} + B e^{-ikx}) + \gamma_1 AB e^{2ikx} + \gamma_2 e^{2\omega t} (A^2 e^{2ikx} + B^2 e^{-2ikx})
+ f^+ e^{\omega t+ikx} + f^- e^{\omega t-ikx} + c.c. + f^m + NRT,
\]

with similar expressions for the remaining fields. The quantities \( f^\pm \) and \( f^m \) represent resonant second order terms, while \( NRT \) denotes non-resonant terms. The superscript \( m \) denotes terms associated with the mean flow; \( f^m \) depends weakly on time but may depend strongly on \( x \). A systematic expansion procedure [119] now leads to the equations

\[
A_t - c_g A_x = \alpha A_{xx} - (\delta + id) A + i(\alpha_3 |A|^2 - \alpha_4 |B|^2) A \\
+ i\alpha_5 \mu B + i\alpha_6 \int_{-1}^{0} g(z)(\psi^m_z)^x dz A + i\alpha_7 (f^m)^x A, 
\] (1.29)

\[
B_t + c_g B_x = \alpha B_{xx} - (\delta + id) B + i(\alpha_3 |B|^2 - \alpha_4 |A|^2) B \\
+ i\alpha_5 \bar{A} - i\alpha_6 \int_{-1}^{0} g(z)(\psi^m_z)^x dz B + i\alpha_7 (f^m)^x B, 
\] (1.30)

\[
A(x + L, t) \equiv A(x, t), \quad B(x + L, t) \equiv B(x, t). 
\] (1.31)

The first seven terms in these equations, accounting for inertia, propagation at the group velocity \( c_g \), dispersion, damping, detuning, cubic nonlinearity and parametric forcing, are familiar from weakly nonlinear, nearly inviscid theories. These theories lead to the expressions

\[
c_g = \omega'(k), \quad \alpha = -\omega''(k)/2, \quad \delta = \alpha_1 C_g^{1/2} + \alpha_2 C_g, \\
\alpha_1 = \frac{k(\omega/2)^{1/2}}{\sinh 2k}, \quad \alpha_2 = \frac{k^2}{4\sigma^2 (1 + 8\sigma^2 - \sigma^4)}, \\
\alpha_3 = \frac{\omega k^2 (1-S)(9-\sigma^2)(1-\sigma^2) + Sk^2(7-\sigma^2)(3-\sigma^2)}{4\sigma^2 [(1-S)\sigma^2 - Sk^2(3-\sigma^2)]}
+ \frac{\omega k^2 [8(1-S) + 5Sk^2]}{4(1-S + Sk^2)}, \\
\alpha_4 = \frac{\omega k^2}{2} \left[ \frac{(1-S+Sk^2)(1+\sigma^2)^2}{(1-S+4Sk^2)\sigma^2} + \frac{4(1-S) + 7Sk^2}{1-S + Sk^2} \right], \quad \alpha_5 = \omega k\sigma,
\]

where \( \omega(k) = [(1-S+Sk^2)k\sigma]^{1/2} \) is the dispersion relation and \( \sigma \equiv \tanh k \). The coefficient \( \alpha_3 \) diverges at (excluded) resonant wave-numbers satisfying \( \omega(2k) = 2\omega(k) \). The detuning \( d \) is given by

\[
d = \alpha_1 C_g^{1/2} - (2\pi NL^{-1} - k)c_g, \quad N = \text{integer},
\]

where the last term represents the mismatch between the wavelength \( 2\pi/k \) selected by the forcing frequency \( 2\omega \) and the domain length \( L \). The last
two terms in equations (1.29)-(1.30) describe the coupling to the mean flow in the bulk (be it viscous or inviscid in origin) in terms of (a local average \( \langle \cdot \rangle \) of) the stream function \( \psi^m \) for this flow and the associated free surface elevation \( f^m \). The coefficients of these terms and the function \( g \) are given by

\[
\alpha_6 = k\sigma/2\omega, \quad \alpha_7 = \omega k(1 - \sigma^2)/2\sigma, \quad g(z) = 2\omega k \cosh[2k(z + 1)]/\sinh^2 k,
\]

(1.32)

and are real. The new terms are therefore conservative, implying that at leading order the mean flow does not extract energy from the system. The mean flow variables in the bulk depend weakly on time but strongly on both \( x \) and \( z \), and evolve according to the equations

\[
\Omega^m = [\psi^m_x + (|A|^2 - |B|^2)g(z)]\Omega^m_x + \psi^m_x \Omega^m_z = C(\Omega^m_{xx} + \Omega^m_{zz}),
\]

\[
\Omega^m = \psi^m_x + \psi^m_z, \quad (1.33)
\]

with

\[
\psi^m_x - f^m_x = \beta_1(|B|^2 - |A|^2)_x, \quad \psi^m_z = \beta_2(|A|^2 - |B|^2),
\]

\[
(1 - S)f^m_x - S \psi^m_{xx} - \psi^m_{zt} + C(\psi^m_{zzz} + 3\psi^m_{xzx}) = -\beta_3(|A|^2 + |B|^2)_x
\]

at \( z = 0 \) \quad (1.34)

and

\[
\psi^m_z = -\beta_4 iAB \ e^{2ikx} + c.c. + |B|^2 - |A|^2, \quad \int_0^L \Omega^m_z \, dx = \psi^m = 0,
\]

at \( z = -1 \) \quad (1.35)

In addition \( \psi^m(x + L, z, t) \equiv \psi^m(x, z, t), \ f^m(x + L, t) \equiv f^m(x, t), \) subject to \( \int_0^L f^m(x, t) \, dx = 0 \), due to mass conservation. Here \( \beta_1 = 2\omega/\sigma, \ \beta_2 = 8\omega k^2/\sigma, \ \beta_3 = (1 - \sigma^2)\omega^2/\sigma^2, \ \beta_4 = 3(1 - \sigma^2)\omega k/\sigma^2 \). Thus the mean flow is forced by the surface waves in two ways. The right sides of the boundary conditions (1.34a) and (1.34c) provide a normal forcing mechanism; this mechanism is the only one present in strictly inviscid theory [61, 144] and does not appear unless the aspect ratio is large. The right sides of the boundary conditions (1.34b) and (1.35a) describe two shear forcing mechanisms, a tangential stress at the free surface [84] and a tangential velocity at the bottom wall [150]. Note that neither of these forcing terms vanishes in the limit of small viscosity (i.e., as \( C \to 0 \)). The shear nature of these forcing terms leads us to retain the viscous term in (1.33a) even when \( C \) is quite small. In fact, when \( C \) is very small, the effective Reynolds number of the mean flow is quite large. Thus the mean flow itself generates additional boundary layers near the top and bottom of the container, and these must be thicker than the original boundary layers for the
validity of the analysis. This puts an additional restriction on the validity of the equations [119]. There is a third, less effective but inviscid, volumetric forcing mechanism associated with the second term in the vorticity equation (1.33)a, which looks like a horizontal force \(|A|^2 - |B|^2)\Omega \Omega^m\) and is sometimes called the vortex force. Although this term vanishes in the absence of mean flow, it can change the stability properties of the flow and enhance or limit the effect of the remaining forcing terms.

Equations (1.29)-(1.31) and (1.33)-(1.35) may be referred to as the general coupled amplitude-mean-flow (GCAMF) equations. Like the equations derived in Section 1.4 they are nonlocal, this time because the parametrically excited waves drive a whole spectrum of almost marginal large scale modes. The GCAMF equations differ from the exact equations forming the starting point for the analysis in three essential simplifications: the fast oscillations associated with the surface waves have been filtered out, the effect of the thin primary viscous boundary layers is replaced by effective boundary conditions on the flow in the bulk, viz. (1.34)b, (1.35)a, and the surface boundary conditions are applied at the unperturbed location of the free surface, viz. \(z = 0\). Thus only the much broader (secondary) boundary layers associated with the (slowly varying) mean flow need to be resolved in any numerical simulation.

The GCAMF equations describe small amplitude slowly varying wave-trains whenever the parameters \(C\), \(L^{-1}\) and \(\mu\) are small, but otherwise unrelated to one another. However, in the distinguished limit

\[
\delta L^2/\alpha = \Delta \sim 1, \quad dL^2/\alpha = D \sim 1, \quad \mu L^2/\alpha \equiv M \sim 1, \tag{1.36}
\]

with \(k = O(1)\) and \(|\ln C| = O(1)\) they can be simplified further, and their relation to the nonlocal equations of Section 1.4 be brought out. These simplified equations are formally valid for \(1 \ll L \ll C^{-1/2}\) if \(k \sim 1\), assuming that \(1 - S \sim 1\). Using \(x\) and \(t\) as fast variables and

\[
X = x/L, \quad \tau = t/L, \quad T = t/L^2 \tag{1.37}
\]

as slow variables, we write

\[
\psi^m(x, z, X, \tau, T) = \psi^s(x, z, X, T) + \psi^i(x, z, X, \tau, T),
\]

\[
\Omega^m(x, z, X, \tau, T) = \Omega^s(x, z, X, T) + \Omega^i(x, z, X, \tau, T),
\]

\[
f^m(x, X, \tau, T) = f^s(x, X, T) + f^i(x, X, \tau, T), \tag{1.38}
\]

and demand that integrals over \(\tau\) of \(\psi^s_x, \psi^s_X, \psi^s, \Omega^s\) and \(f^s\) be bounded as \(\tau \to \infty\), i.e., that the nearly inviscid mean flow is purely oscillatory on the timescale \(\tau\). In terms of these variables the local horizontal average <(\gamma) becomes an average over the fast variable \(x\). Since its frequency is of order \(L^{-1}\) (see equation (1.37)), which is large compared with \(C\), the inertial term for this flow is large in comparison with the viscous terms (see
equation (1.33)), except in two secondary boundary layers, of thickness of order \((CL)^{1/2} \ll 1\), attached to the bottom plate and the free surface. Note that, as required for the consistency of the analysis, these boundary layers are much thicker than the primary boundary layers associated with the surface waves, which provide the boundary conditions (1.34)-(1.35) for the mean flow. Moreover, the width of these secondary boundary layers remains small as \(\tau \to \infty\) and (to leading order) the vorticity of this nearly inviscid mean flow remains confined to these boundary layers. It is possible to check that these boundary layers do not modify the boundary conditions (1.34)-(1.35) on the nearly inviscid bulk flow.

The complex amplitudes and the flow variables associated with the nearly inviscid bulk flow may be expanded as

\[
(A, B) = L^{-1}(A_0, B_0) + L^{-2}(A_1, B_1) + \ldots,
\]

\[
(\psi^\prime, \Omega^\prime, f^\prime) = L^{-2}(\phi_{0,0}^0, W_0^0, 0) + L^{-3}(\phi_{0,1}^0, W_1^0, F_0^0) + \ldots,
\]

\[
(\psi^a, \Omega^a, f^a) = L^{-2}(\phi_{0,0}^a, 0, F_0^a) + L^{-3}(\phi_{0,1}^a, W_1^a, F_1^a) + \ldots \tag{1.39}
\]

Substitution of (1.36)-(1.39) into (1.29)-(1.35) leads to the following:

(i) From (1.33)-(1.35), at leading order,

\[
\phi_{0,xx} + \phi_{0,zz} = 0 \quad \text{in} \quad -1 < z < 0, \quad \phi_{0}^i = 0 \text{ at } z = -1, \quad \phi_{0,x}^i = 0 \text{ at } z = 0,
\]

together with \(F_{0,x}^i = 0\). Thus

\[
\phi_{0}^i = (z + 1)\Phi_{0}^i(X, \tau, T), \quad F_{0}^i = F_{0}^i(X, \tau, T).
\]

At second order, the boundary conditions (1.34)a,c yield

\[
\phi_{0,x}(x, 0, X, \tau, T) = F_{0,x}^i - \Phi_{0,x}^i + \beta_1(|B_0|^2 - |A_0|^2)x,
\]

\[
(1 - S)F_{0,x}^i - SF_{0,x}^i = \Phi_{0,\tau}^i - (1 - S)F_{0,x}^i - \beta_3(|A_0|^2 + |B_0|^2)x
\]

at \(z = 0\). Since the right hand sides of these two equations are independent of the last variable \(x\) and both \(\phi_{0}^i\) and \(F_{0}^i\) must be bounded in \(x\), it follows that

\[
\Phi_{0,X}^i - F_{0,\tau}^i = \beta_1(|B_0|^2 - |A_0|^2)x, \quad \Phi_{0,\tau}^i - c_p F_{0,X}^i = \beta_3(|A_0|^2 + |B_0|^2)x, \tag{1.40}
\]

where

\[
c_p = (1 - S)^{1/2} \tag{1.41}
\]

is the phase velocity of long wavelength surface gravity waves. Equations (1.40) must be integrated with the following additional conditions

\[
\Phi_{0}^i(X + 1, \tau, T) \equiv \Phi_{0}^i(X, \tau, T), \quad F_{0}^i(X + 1, \tau, T) \equiv F_{0}^i(X, \tau, T),
\]

and the requirements that integrals over \(\tau\) of \(\Phi_{0,X}^i\) and \(F_{0}^i\) remain bounded as \(\tau \to \infty\), with \(\int_0^1 F_{0}^i \, dX = 0\).
(ii) The leading order contributions to equations (1.29)-(1.30) yield $A_0 - c_p A_0 X = B_0 - c_p B_0 X = 0$. Thus $A_0 = A_0(\xi, T), B_0 = B_0(\eta, T)$, where $\xi = X + c_p \tau$ and $\eta = X - c_p \tau$ are again the characteristic variables, and $A_0(\xi + 1, T) \equiv A_0(\xi, T), B_0(\eta + 1, T) \equiv B_0(\eta, T)$.

Substitution of these expressions into (1.40) followed by integration of the resulting equations yields

$$
\Phi_i^0 = \frac{\beta_1 c_p^2 + \beta_3 c_p}{c_p^2 - c_p^2} [\langle |A_0|^2 \rangle - |B_0|^2 - \langle |A_0|^2 - |B_0|^2 \rangle X] + c_p [F^+(X + c_p \tau, T) - F^-(X - c_p \tau, T)],
$$

$$
F_0^i = \frac{\beta_1 c_p + \beta_3}{c_p^2 - c_p^2} [\langle |A_0|^2 \rangle + |B_0|^2 - \langle |A_0|^2 + |B_0|^2 \rangle X] + F^+(X + c_p \tau, T) + F^-(X - c_p \tau, T)],
$$

where $\langle \cdot \rangle^X$ denotes the mean value in the slow spatial variable $X$, and the functions $F^\pm$ are such that

$$
F^\pm(X + 1 \pm c_p \tau, T) \equiv F^\pm(X \pm c_p \tau, T), \quad \langle F^\pm \rangle^X = 0.
$$

The particular solution of (1.42)-(1.43) yields the usual inviscid mean flow included in nearly inviscid theories [144]; the averaged terms are a consequence of volume conservation [144] and the requirement that the nearly inviscid mean flow has a zero mean on the timescale $\tau$; the latter condition is never imposed in strictly inviscid theories but is essential in the limit we are considering, as explained above. To avoid breakdown of the solution (1.42)-(1.43) at $c_p = c_p$ we assume that

$$
|c_p - c_p| \sim 1.
$$

(iii) The evolution equations for $A_0$ and $B_0$ on the timescale $T$ are readily obtained from equations (1.29)-(1.31), invoking (1.36), (1.38) and (1.42)-(1.44), and eliminating secular terms (i.e., requiring $|A_1|$ and $|B_1|$ to be bounded on the timescale $\tau$):

$$
A_0 T = i\alpha A_0 \xi - (\Delta + i D) A_0 + i[\alpha_3 + \alpha_8]|A_0|^2 - \alpha_8 \langle |A_0|^2 \rangle^X
$$

$$
- \alpha_4 \langle |B_0|^2 \rangle^\eta A_0 + i\alpha_5 M \langle \overline{B_0} \rangle^\eta + i\alpha_6 \int_{-1}^{0} g(z) \langle \phi_0 \rangle^x \phi z \, dz A_0,
$$

$$
B_0 T = i\alpha B_0 \eta - (\Delta + i D) B_0 + i[\alpha_3 + \alpha_8]|B_0|^2 - \alpha_8 \langle |B_0|^2 \rangle^\eta
$$

$$
- \alpha_4 \langle |A_0|^2 \rangle^\xi B_0 + i\alpha_5 M \langle \overline{A_0} \rangle^\xi - i\alpha_6 \int_{-1}^{0} g(z) \langle \phi_0 \rangle^z \phi z \, dz B_0.
$$

Here $\langle \cdot \rangle^x, \langle \cdot \rangle^X, \langle \cdot \rangle^\xi$ and $\langle \cdot \rangle^\eta$ denote mean values over the variables $x, X, \xi$ and $\eta$, respectively, and

$$
\alpha_8 = \frac{[\alpha_6(2\omega/\sigma)(\beta_1 c_p^2 + \beta_3 c_p) + \alpha_7(\beta_1 c_p + \beta_3)]/(c_p^2 - c_p^2)}.
$$
Equations (1.46) are independent of $F^\pm$ because of the second condition in (1.44).

When $\Delta > 0 \, (|A_0|^2 - |B_0|^2)^\tau = (|A_0|^2)^\xi - (|B_0|^2)^\eta \to 0$ as $T \to \infty$, and equations 1.33-1.35 become

$$W_0^v - \phi_{0z}^v W_0^v + \phi_{0x}^v W_0^v = Re^{-1}(W_{0xx}^v + W_{0zz}^v), \quad \phi_{0xx}^v + \phi_{0zz}^v = W_0^v,$$

in $-1 < z < 0$, \quad (1.47)

$$\phi_{0x}^v = \phi_{0zz}^v = 0 \text{ at } z = 0,$$

$$(W_{0x}^v)^x = \phi^v_0 = 0, \quad \phi^v_0 = -\beta_4[i(A_0 B_0)^\tau e^{2i\pi x} + c.c.]$$

at $z = -1$, \quad (1.49)

$$\phi_0^v(x + L, x + 1, z, T) \equiv \phi_0^v(x, X, z, T),$$

(1.50)

where

$$Re = 1/(CL^2)$$

is the effective Reynolds number of the viscous mean flow. Note that $Re = O(C^{-1/2})$ if $k \sim 1$.

Some remarks about these equations and boundary conditions are now in order.

a. The viscous mean flow is driven by the short gravity-capillary waves through the inhomogeneous term in the boundary condition (1.49)c, i.e., at the lower boundary. Since $(A_0 B_0)^\tau$ depends on both $X$ and $T$ (unless either $A_0$ or $B_0$ is spatially uniform) the boundary condition implies that $\phi_0^v$ (and hence $W_0^v$) depends on both the fast and slow horizontal spatial variables $x$ and $X$. This dependence cannot be obtained in closed form, and one must therefore resort to numerical computations for realistically large values of $L$. Note, however, that in fully three-dimensional situations [68] in which lateral walls are included a viscous mean flow will be present even when $k \gg 1$ because the forcing of the mean flow in the oscillatory boundary layers along these walls remains.

b. More importantly the change of variables

$$A_0 = \tilde{A}_0 \, e^{-i k \theta}, \quad B_0 = \tilde{B}_0 \, e^{i k \theta},$$

(1.52)

reduces equations (1.46) to the much simpler form

$$\tilde{A}_0 T = i \alpha \tilde{A}_0 \xi - (\Delta + i D) \tilde{A}_0 + i[(\alpha_3 + \alpha_8)|\tilde{A}_0|^2 - (\alpha_4 + \alpha_8)|\tilde{B}_0|^\eta]^\tau \tilde{A}_0 + i\alpha_5 M(\tilde{B}_0)^\eta,$$

$$\tilde{B}_0 T = i \alpha \tilde{B}_0 \eta - (\Delta + i D) \tilde{B}_0 + i[(\alpha_3 + \alpha_8)|\tilde{B}_0|^2]^\tau \tilde{B}_0 + i\alpha_5 M(\tilde{A}_0)^\xi,$$

$$\tilde{A}_0(\xi + 1, T) \equiv \tilde{A}_0(\xi, T), \quad \tilde{B}_0(\eta + 1, T) \equiv \tilde{B}_0(\eta, T).$$

(1.54)
from which the mean flow is absent. This decoupling is a special property of the regime (1.36), but is not unique to it. The resulting equations provide perhaps the simplest description of the Faraday system at large aspect ratio, and it is for this reason that they have been extensively studied [86, 132]. Except for the presence of parametric forcing term $M$ they are of the same form as the coupled nonlocal CGL equations derived in Section 1.4. Tracing back through the derivation one can see that the nonlocal terms in both sets of equations have identical origin: dynamics occurs on two distinct slow timescales.

In short domains, on the other hand, with $L = O(1)$ only two timescales are present, and the resulting coupled amplitude-mean flow equations are considerably simpler:

$$
A_t = - (\delta + id) A + i(\alpha_3|A|^2 - \alpha_4|B|^2)A + i\alpha_5 \mu \tilde{B} - i\alpha_6 L^{-1} \int_{-1}^{0} \int_{0}^{L} g(z) u^z \, dx \, dz A,
$$

$$
B_t = - (\delta + id) B + i(\alpha_3|B|^2 - \alpha_4|A|^2)B + i\alpha_5 \mu \tilde{A} + i\alpha_6 L^{-1} \int_{-1}^{0} \int_{0}^{L} g(z) u^z \, dx \, dz B,
$$

(1.55)

with $A$ and $B$ spatially constant and the coefficients given by expressions that are identical to those in (1.29)-(1.30). Here $t$ denotes a slow time whose magnitude is determined by the damping $\delta > 0$ and the detuning $d$, both assumed to be of the same order as the forcing amplitude $\mu$; in the long time limit $|A|^2 = |B|^2 \equiv R^2$. It follows that the mean flow $(u^r(x,z,t), u^\theta(x,z))$ is now entirely viscous in origin, and obeys a two-dimensional Navier-Stokes equation of the form (1.47). If we absorb the standing wave amplitude $R$ (and some other constants) in the definition of the Reynolds number

$$
Re \equiv 2 \beta_4 R^2 / C_g
$$

(1.56)

this equation is to be solved subject to the boundary conditions

$$
u^r = - \sin(2k(x - \theta)), \quad u^\theta = 0 \quad \text{at} \quad z = -1,
$$

$$
u^z = 0, \quad u^\theta = 0 \quad \text{at} \quad z = 0.
$$

(1.57)

Because the structure of equations (1.55) is identical to that of equations (1.46) the change of variables (1.52) leads to a decoupling of the amplitudes from the spatial phase $\theta$ which now satisfies the equation

$$
\theta_t = \frac{\alpha_6}{kL} \int_{-1}^{0} \int_{0}^{L} g(z) \nu^r(x, z, t) \, dx \, dz.
$$

(1.58)

Martín et al [8] solve the resulting equations numerically with periodic boundary conditions (corresponding to an annular domain of length $L$)
and demonstrate that a basic pattern of subharmonic standing waves can indeed lose stability at finite amplitude through the excitation of a viscous mean flow. This may happen in one of two ways, either at a parity breaking steady state bifurcation or at a reflection symmetry-breaking Hopf bifurcation. The former results in a steady drift of the pattern while the latter produces a state that has been called a direction-reversing wave [7].

1.6 Nonlinear waves in extended systems with broken reflection symmetry

The effects of distant boundaries on the onset of travelling wave instabilities in systems with broken reflection symmetry differ remarkably from the corresponding steady-state situation in reflection-symmetric systems. In the latter case the imposition of boundary conditions at the ends of a domain of aspect ratio \( L \) leads to a correction to the instability threshold of \( O(L^{-2}) \) when \( L \) is large. However, as mentioned already in the introduction, in systems that lack reflection symmetry the primary instability is always a Hopf bifurcation to travelling waves with a preferred direction of propagation, and in this case the imposition of similar boundary conditions results in an \( O(1) \) change to the threshold for instability. Moreover, the initial eigenfunction at the onset is a wall mode rather than wave-like, and the frequency of this mode differs substantially from that of the most unstable, spatially periodic solution. With increasing values of the instability parameter \( \mu \) a front forms, separating an exponentially small wave-train near the upstream boundary from a fully developed one downstream, with a well-defined wave-number, frequency and amplitude, whose location moves further and further upstream as \( \mu \) continues to increase. Both the spatial wave-number and the amplitude are controlled by the temporal frequency, which is in turn controlled by \( \mu \), and in the fully nonlinear regime by the boundary conditions as well. The resulting changes in the frequency may trigger secondary transitions to quasiperiodic and/or chaotic states. These phenomena have been described in a recent paper by Tobias et al [156] (see also [54]), and the criterion for the presence of a global unstable mode was found to be closely related to that for absolute instability of the basic state in an unbounded domain. The secondary transitions are likewise related to absolute instability of the primary wave-train.

The simplest model of this set of phenomena is provided by the complex Ginzburg-Landau equation with drift:

\[
\frac{\partial A}{\partial t} = c_g \frac{\partial A}{\partial x} + \mu A + a|A|^2 A + \lambda \frac{\partial^2 A}{\partial x^2}, \quad 0 \leq x \leq L ,
\]

subject to the boundary conditions

\[
A(0) = A(L) = 0.
\]
Here $c_g > 0$ represents the drift term (and is the group velocity $d\omega/dk$ if $A(x,t)$ is the complex amplitude of a wave train with frequency $\omega(k)$) and
\[ \lambda = 1 + i\alpha, \quad a = -1 + i\beta, \]
In the following we take $c_g$ to be of order unity, so that reflection symmetry is broken strongly. In figure 1.6 we show a solution of equations (1.59)-(1.60) obtained numerically for $\mu > \mu_0 \equiv c_g^2/[4(1 + \lambda^2)]$ and $c_g = 1$. The figure is in the form of a space-time diagram, with time increasing upwards. Infinitesimal disturbances travel to the left from $x = L$ and grow as they do so, evolving into a stationary primary front, separating a small amplitude upstream wave (barely visible) from a fully developed downstream wave with $O(1)$ amplitude. For $\mu$ not too large this wave-train persists until almost $x = 0$, where there is a boundary layer within which $A$ drops to zero. However, for larger $\mu$ a remarkable phenomenon appears. The wave-train develops a secondary front prior to $x = 0$ separating the laminar $O(1)$ wave-train, and a new wave-train in the region adjacent to $x = 0$. In the example shown this wave-train is chaotic or ‘turbulent’. As a result the position of the secondary front fluctuates but nonetheless has a well-defined mean. For other parameter values one may find a secondary front between two periodic wave-trains, with different amplitudes and wave-numbers, and if there are phase slips along the front, different frequencies as well.

Figure 1.7 shows another example, this time computed from equations describing the mean field dynamo [147]
\[
\frac{\partial A}{\partial t} = \frac{DB}{1 + B^2} + \frac{\partial^2 A}{\partial x^2} - A, \quad \frac{\partial B}{\partial t} = \frac{\partial A}{\partial x} + \frac{\partial^2 B}{\partial x^2} - B. \tag{1.61}
\]
Here $D > 0$ is a bifurcation parameter analogous to $\mu$, and $A$ and $B$ are the poloidal field potential and the toroidal field itself. Both are real-valued functions. This model has the merit of describing a real magnetic field as opposed to the CGL equation, which only describes the evolution of a slowly varying envelope of a short, high-frequency wave-train. The figure is computed for (a) $D = 8.75$ and (b) $D = 15.0$, subject to the boundary conditions $A_{x} = B = 0$ at $x = 0$ and $A = B = 0$ at $x = L$. Figure 1.7(a) shows a single front near the right boundary separating a stationary fully-developed but laminar wave-train from a region with an exponentially small state near $x = L$. In contrast, figure 1.7(b) shows two fully developed wave-trains with well-defined wave-numbers, amplitudes and frequencies separated by a secondary front in the left half of the container. Both wave-trains are laminar and are separated by a more-or-less stationary secondary front. It is tempting to think of these fronts as shock-like structures, across which there is finite jump in three quantities: amplitude, wave-number and perhaps frequency as well. These shocks are of considerable mathematical interest since the underlying equations are neither hyperbolic nor conservative. As a result there are no obvious candidates for the type of jump conditions familiar from gas dynamics. The a
priori prediction of the location of these shocks, and of the associated jump conditions as a function of $\mu$ constitutes an interesting and important problem for future research. Analysis of the CGL equation (1.59) shows that these secondary fronts form once the primary wave train loses stability to a secondary absolute Benjamin-Feir instability [156]. As already mentioned the resulting scenario also arises in attempts to explain the far-field breakup of spiral waves in reaction-diffusion systems [155].

As is well known systems that are only convectively unstable (but absolutely stable) act as efficient noise amplifiers [69]. This property man-
Figure 1.7. Colour-coded space-time plots obtained for the dynamo equations (1.61) with $L = 300.0$ and (a) $D = 8.75$, (b) $D = 15.0$. The domain $(0, L)$ lies along the horizontal axis, with time increasing upward. In (a) the primary front selects the amplitude and wave-number of the finite amplitude dynamo waves; in (b) a secondary front separates the primary wave train on the right from the secondary wave-train on the left.

Itself in extended but finite domains as well, as discussed by Proctor et al [148]. Figure 1.8 summarizes the effects of injecting small amplitude noise at the upstream boundary in the two cases of interest: for Benjamin-Feir stable ($\lambda_{RF} < 1$) and Benjamin-Feir unstable ($\lambda_{RF} > 1$) regimes. In the former case the spatially uniform Stokes solution of (1.59) is stable with respect to long wavelength perturbations; this is not the case in the Benjamin-Feir unstable regime. As a result any nontrivial dynamics present in the system must be a consequence of the boundaries. Little changes if the noise is injected instead throughout the domain $0 < x < L$,.
since the dynamics is primarily determined by the perturbations that have travelled furthest, and hence grew the most.

If the trivial state \( A = 0 \) is completely stable, then all such disturbances decay at every location, and so have little effect. However, if the trivial state is convectively unstable, there are disturbances that grow in an appropriately moving reference frame, even though at fixed location all disturbances eventually decay. In these circumstances the continued injection of disturbances at an upstream location can produce persistent noise-sustained structures downstream [69]. Such structures have been studied in experiments [50, 49, 114], and in various model problems based on the complex Ginzburg-Landau equation [52, 69, 70, 71, 72, 85, 97, 104]. In fact the addition of noise can lead to the destabilization not only of the trivial state but also of the primary wave-train (figure 1.8). Moreover the travelling wave nature of the transient instability that is sustained by noise injection leads to a powerful frequency selection effect, which determines the spatio-temporal properties of the resulting noise-sustained structures. This mechanism, identified in [148] in the context of the CGL equation (1.59), is believed to have general applicability. Briefly, the system selects the frequency \( \omega_{\text{max}} \) that maximizes the smallest spatial growth rate in the downstream direction. This frequency selection mechanism differs from the intuitive (but incorrect) idea that one wishes to maximize the downstream spatial growth rate on the grounds that the mode growing most rapidly to the left would be the one observed. This is a consequence of the downstream boundary condition. Because of this boundary the solution of the linear problem on \( 0 < x < L \) consists of a linear combination of exponentials involving the (complex) roots of the dispersion relation. At the left (or downstream) boundary the amplitudes of all the component modes are in general of the same order; for each value of the frequency \( \omega \) the mode seen far from the boundary is then the one that decays least rapidly to the right. One seeks therefore the frequency \( \omega_{\text{max}} \) that maximizes this least decay rate. Figure 1.9 provides a confirmation of this idea for the dynamo equations (1.61). Thus once again the mere fact that there is a downstream boundary has a profound effect on the behaviour of the system: open (i.e., semi-infinite) systems without a downstream boundary behave very differently, and for these the intuitive argument is correct. It is worth mentioning that in the absence of noise a semi-infinite system without a downstream boundary only generates structure above the absolute instability threshold. In contrast a semi-infinite system without an upstream boundary is unstable already above the convective instability threshold \( \mu = 0 \) [120].

The linear theory just described predicts [148] that the noise-sustained structure will be present in \( x < x_{\text{front}} \), where

\[
x_{\text{front}} \equiv L - \kappa_{\text{max}}^{-1} \ln \epsilon.
\]  

(1.62)

Here \( \kappa_{\text{max}} \) is the maximum value of the least spatial growth rate determined
by the above procedure, and \( \varepsilon \) is a measure of the amplitude of the noise injected at \( x = L \), assumed to be uniformly distributed in \( [-\varepsilon, \varepsilon] \). For (1.59) \( \omega_{\text{max}} = \mu \lambda_f \) and the resulting noise-sustained structure has a well-defined wave-number and amplitude, given by

\[
k = \frac{1}{2(a_I + \lambda_f)} \left( c_g - \sqrt{c_g^2 - 4\mu(\lambda_f^2 - \alpha_f^2)} \right),
\]

\[
|A_0|^2 = \frac{1}{2(a_I + \lambda_f)^2} \left( 4\mu \lambda_f(a_I + \lambda_f) - c_g^2 + c_g \sqrt{c_g^2 - 4\mu(\lambda_f^2 - \alpha_f^2)} \right).
\]

(1.63)

As shown in [148] this solution can in turn be convectively unstable, and so support secondary noise-induced structures, or be absolutely unstable (cf. figure 1.8).

The phenomenon of a noise-sustained secondary instability is perhaps of greatest interest. We consider therefore the case of a primary wave-train when \( \mu_f < \mu < \mu_f^{\text{sec}} \), i.e., when this wave-train is convectively unstable. The primary wave-train is unaffected by noise injection at the upstream boundary because it is a global mode (\( \mu > \mu_f = \mu_a + O(L^{-2}) \)). In the absence of noise the secondary convective instability has no effect, provided that the group velocity associated with this instability is leftward. The argument is the same as for the primary instability: perturbations are advected towards the boundary at \( x = 0 \) where they are dissipated, and the instability will produce secondary structures only when the threshold for secondary absolute instability is exceeded. However, in the presence of noise a noise-sustained secondary structure will form. This structure appears first the left boundary, and with increasing \( \mu \) extends further and further to the right, separated from the primary wave-train by a secondary front. Figure 1.10 shows an example computed from equation (1.59) for \( \mu = 1, \varepsilon = 10^{-4}, \lambda_f = 0.43, a_f = 2 \) and \( c_g = 1 \). For these parameters the secondary instability boundary in the absence of noise is at \( \mu_a^{\text{sec}} \approx 1.5 \), and indeed the chaotic structure present near the left boundary in figure 1.10 decays away when \( \varepsilon \) is set to zero. Guided by the discussion of the primary instability we expect that in its linear phase the secondary mode has a well-defined spatial growth rate, wave-number and temporal frequency, and that these are determined by a criterion similar to that for the primary noise-sustained mode. This expectation is borne out by an investigation of the linear instability of the primary wave-train along the lines of [156].

### 1.7 Summary and conclusions

In this article I have pointed to a number of issues in the general area of pattern formation which remain unsolved, and attempted to indicate why I believe they are important. The problems mentioned range from
the mathematical (extension of KAM theory into the spatial domain) to the practical, such as confronting the predictions of model equations with experimental data, or at least data obtained from accurate numerical simulations of the governing field equations. I discussed various techniques for generating such model equations, including both rigorous derivations and symmetry-based methods. In each case I discussed the assumptions required and their appropriateness for the physical problem at hand. In many cases these equations provide us with a good but partial understanding; in some problems, such as the oscillons in granular media, this may be as much as we can expect, while in others much more can in principle be done. This is because in the former case we know only the microscopic interactions but lack effective constitutive relations for a macroscopic theory, while in other areas the correct field equations are well known. Many pattern forming systems, for example chemical systems, fall in between these two extremes. These systems are governed by an ever growing variety of model equations whose structure depends sensitively on the assumptions made about the spatial and temporal scales in the system, both intrinsic and externally imposed. In this article I have emphasized systems that are described by fundamentally nonlocal equations and explained the reasons for this fact. These equations provide new classes of pattern-forming equations and their behaviour is only just beginning to be understood. Traditional equivariant theory is of much less use in these systems because of its implicit assumptions that all interactions are local, that is, local in both space and time, and in phase space. As a result the derivations may appear to be involved, but at present this is unavoidable. I have also emphasized the importance of considering the applicability or relevance of the solutions of these model equations to the underlying physical system. These questions raise additional mathematical issues, some well known from the theory of averaging, but others such as imperfection sensitivity of equivariant dynamics still poorly understood.

I expect substantial progress on a number of the issues raised over the next several years, although I will not risk making any predictions. The field of pattern formations is a lively one, and this article will have succeeded if it stimulates both new work and new ideas in this area.

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<th>Name</th>
<th>$\text{Fix}(\Sigma)$</th>
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<td>0.   Trivial Solution ($T$)</td>
<td>$z = (0, 0, 0, 0)$</td>
<td>$\Gamma$</td>
</tr>
<tr>
<td>I.   Travelling Rolls ($TR$)</td>
<td>$z = (v_1, 0, 0, 0)$</td>
<td>$SO(2) \times SO(2)$</td>
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<tr>
<td>II.  Standing Rolls ($SR$)</td>
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<td>$Z_2 \times SO(2)$</td>
</tr>
<tr>
<td>III. Standing Squares ($SS$)</td>
<td>$z = (v_1, v_1, v_1, v_1)$</td>
<td>$SO(2)$</td>
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<tr>
<td>IV.  Alternating Rolls ($AR$)</td>
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</tr>
<tr>
<td>V.   Standing Cross-rolls ($SCR$)</td>
<td>$z = (v_1, v_2, v_1, v_2)$</td>
<td>$Z_2$</td>
</tr>
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**Table 1.1.** Fixed point subspaces corresponding to the different isotropy subgroups $\Sigma$ of $\Gamma$. The coordinates specifying each $\text{Fix}(\Sigma)$ are given terms of $z = (z_1, z_2, z_3, z_4)$ with each $v_j \in \mathbb{C}, j = 1, 2$. 
**Figure 1.8.** Schematic diagram of the possible roles of noise in the CGL equation (1.39). The two scenarios presented are for (a) Benjamin-Feir stable parameter values ($\lambda_1 a_1 < 1$) and (b) Benjamin-Feir unstable parameter values ($\lambda_1 a_1 > 1$). In (a) the role of the noise depends critically on the value of the bifurcation parameter, while in (b) the noise-induced primary wave-train is always susceptible to noise-induced disruption as discussed by Deissler [70].
Figure 1.9. Noise-sustained dynamo waves computed from equations (1.61): selected frequency $\omega$ as a function of the dynamo number $D$ in the convectively unstable regime $D_c < D < D_\alpha$. The solid line is the boundary of the region in which wave-trains occur. The dashed line shows the theoretical prediction. The symbols show the results of numerical simulations with different values of $D$ and noise level $\epsilon$. Dotted lines show thresholds for convective and absolute instability of the trivial state.
Figure 1.10. Noise-sustained secondary structures in the CGL equation (1.59). Space-time plot of $\text{Re}(A)$ for $\mu = 1.0$, $\epsilon = 10^{-4}$, $\lambda_I = 0.45$ and $\alpha_I = 2.0$ with $x$ increasing to the right and time increasing upwards. The irregular phase at the left of the picture is sustained by noise injection at the right boundary, and decays away in its absence.