New types of waves in systems with $O(2)$ symmetry

A.S. Landsberg and E. Knobloch

Department of Physics, University of California at Berkeley, Berkeley, CA 94720, USA

Received 28 April 1993; accepted for publication 9 June 1993
Communicated by D.D. Holm

A number of novel phenomena arising in systems with $O(2)$ symmetry are described. These include pulsing traveling waves with either a periodically or quasiperiodically modulated phase velocity, and heteroclinic waves which alter their appearance with time, taking successively the form of pure traveling waves, standing waves and steady states. Chaotic waves can also be present. These states result from the codimension-two interaction between a reflection-breaking steady state bifurcation and a reflection-preserving Hopf bifurcation from a circle of nontrivial equilibria. The resulting waveforms are illustrated using a model of magnetoconvection with periodic boundary conditions.

1. Introduction

Dynamical systems with $O(2)$ symmetry may have two types of equilibria, the trivial equilibrium that is invariant under $O(2)$ and nontrivial equilibria that break $O(2)$ symmetry. The latter are taken into one another by rotations $R_{\phi}\in O(2)$ and constitute a group orbit of equilibria. Convection with periodic boundary conditions on a line provides a good example of both types of equilibria: the conduction state is a trivial equilibrium, while the convection state is a nontrivial equilibrium in the above sense. In this example translations in the horizontal mod the spatial period are identified with the rotations $R_{\phi}\in O(2)$.

As a parameter is varied a branch of nontrivial equilibria typically bifurcates from the trivial equilibrium. These nontrivial equilibria are reflection symmetric [1], i.e., they are invariant under the reflection $x\in O(2)$. This paper is concerned with secondary bifurcations from this circle of nontrivial equilibria. There are two types of such bifurcations, those that preserve the reflection symmetry and those that break it. In the former case one generically finds simple saddle-node bifurcations or Hopf bifurcations to standing oscillations. In the latter case one finds steady state bifurcations to traveling waves (either left- or right-going) or Hopf bifurcations to direction-reversing waves [2].

In the present work we study the interaction of a reflection-preserving Hopf bifurcation with a reflection-breaking steady state bifurcation. This codimension-two bifurcation describes the interaction of standing oscillations and traveling waves in a system near the circle of nontrivial equilibria. We construct below the normal form for this interaction and show that in its unfolding one finds a variety of novel dynamical behavior, including pulsing traveling waves, three-tori and heteroclinic waves. We illustrate these solutions by explicit examples computed from a model of magnetoconvection. By breaking the normal form symmetry chaotic waves can be produced.

2. Normal form equations and their solutions

The dynamics at the codimension-two bifurcation takes place on the corresponding center manifold. This manifold is four-dimensional, two dimensions coming form the Hopf bifurcation to standing waves and two form the steady state bifurcation to traveling waves. This steady state bifurcation is two-dimensional since the nontrivial equilibrium has an “extra” zero eigenvector out of the reflection-invariant subspace, corresponding to neutral stability under rotations (translations). In appropriate coordinates, the resulting equations take the form
\[
\begin{align*}
\frac{dx}{dt} &= -Qy + F_1(x, y, z^2), \tag{1a} \\
\frac{dy}{dt} &= Qx + F_2(x, y, z^2), \tag{1b} \\
\frac{dz}{dt} &= zG(x, y, z^2), \tag{1c} \\
\frac{d\phi}{dt} &= zH(x, y, z^2). \tag{1d}
\end{align*}
\]

Here \((x, y)\) are the coordinates in the reflection-invariant subspace, and describe standing oscillations in the system; \(Q\) is the corresponding Hopf frequency. A nonzero \(z\) indicates that reflection symmetry is broken, and implies that the resulting pattern drifts with phase velocity \(\frac{d\phi}{dt}\). In particular \(d\phi/dt = 0\) in the reflection-invariant subspace \(z=0\).

The origin \((x=y=z=0, \phi\) arbitrary) corresponds to the circle of nontrivial equilibria; the invariance of the equations with respect to \((z, \phi) \rightarrow (-z, -\phi)\) is a consequence of the reflection symmetry \(\kappa\) of these nontrivial equilibria.

Equations (1) can be put into normal form by a series of near-identity nonlinear coordinate changes equivariant with respect to \(\kappa\). Truncating the resulting equations at third order one obtains

![Fig. 1. A representative bifurcation diagram for eqs. (3a), (3b) for \(A>0, C>0, E<0, F<0, AF-EC>0\). In region I, the traveling wave (TW) is stable, while the nontrivial equilibrium (SS) and standing wave (SW) are both unstable. In region II, the TW has become unstable, giving birth to a pulsating traveling wave (PW). In region III the PW has lost stability through a Hopf bifurcation, producing a new pulsating wave (represented by a limit cycle in \((\rho, z)\) space) whose phase velocity oscillates quasiperiodically. As the bifurcation parameters vary this oscillation grows, and a heteroclinic cycle forms connecting the SS, TW and SW.](image-url)
\[ \frac{dx}{dt} = -\Omega y + (Ax - By)(x^2 + y^2) \]
\[ + (Cx - Dy)z^2, \quad (2a) \]
\[ \frac{dy}{dt} = \Omega x + (Bx + Ay)(x^2 + y^2) \]
\[ + (Dx + Cy)z^2, \quad (2b) \]
\[ \frac{dz}{dt} = Ez(x^2 + y^2) + Fz^2, \quad (2c) \]
\[ \frac{d\phi}{dt} = z[A' + B'x + C'y \]
\[ + D'(x^2 + y^2) + E'z^2]. \quad (2d) \]

Here \( A, B, \ldots, E' \) are normal form coefficients, and will be taken as known (nonzero) constants in what follows. The truncation at third order can be justified under appropriate nondegeneracy conditions on these coefficients. In polar coordinates defined by \( \rho^2 = x^2 + y^2, \quad \theta = \arctan(y/x) \) the unfolded normal form of these equations is

\[ \frac{d\rho}{dt} = \rho(v_1 + Ap^2 + Cz^2), \quad (3a) \]
\[ \frac{dz}{dt} = z(v_2 + Er^2 + Fz^2), \quad (3b) \]
\[ \frac{d\theta}{dt} = \Omega + Bp^2 + Dz^2, \quad (3c) \]
\[ \frac{d\phi}{dt} = z(A' + B'\rho \cos \theta + C'\rho \sin \theta \]
\[ + D'\rho^2 + E'z^2). \quad (3d) \]

Here \( v_1 \) and \( v_2 \) are the two unfolding parameters; the codimension-two bifurcation occurs at \( v_1 = v_2 = 0 \). Observe that in normal form both phases decouple from the remaining equations. Equations \((3a)-(3c)\) are the standard normal form for the interaction of

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{A pulsing traveling wave (PW) calculated from normal form equations \((3a)-(3d)\) using the representation \((4), (5), valid at mid-layer. This two-frequency wave is characterized by an oscillatory amplitude and phase velocity.}
\end{figure}
Hopf and pitchfork bifurcations [3]. The possible solutions are therefore already known (see fig. 1). Their interpretation is, however, modified because of eq. (3d): the solution $(0,0)$ corresponds to the nontrivial equilibria, while fixed points $(\rho,0)$, $(0,z)$ and $(\rho,z)$ correspond, respectively, to a standing
A pure traveling wave and a pulsing traveling wave. In the following we refer to these solutions as SS, SW, TW and PW, respectively. The PW is a two-frequency wave; owing to the rotational invariance of the system no frequency locking can take place, however. We use here the descriptive term pulsing traveling wave to distinguish this modulated wave from the modulated traveling waves present, for example, in the Takens–Bogdanov bifurcation with $O(2)$ symmetry [4,5] and from the direction-reversing traveling waves (RW) studied by Landsberg and Knobloch [2] (called pulsating waves by Proctor and Weiss [6]). In a PW the phase velocity executes (small) oscillations about a nonzero mean, and hence, in contrast to the RW, a PW propagates in one direction only, its velocity “pulsing” as the wave propagates. In fig. 2 we illustrate this behavior using the representation (cf. ref. [2])

$$\psi(x, t) = \text{Re}[a(t)e^{ikx}],$$

where $\psi(x, t)$ is the (scalar) field of interest, characterized by wavenumber $k$ and amplitude $a(t) = [a_0 + \rho(t) \cos \theta(t)]e^{i\phi(t)} \ldots (5)$

As a function of the two unfolding parameters $\nu_1$ and $\nu_2$, the SW and TW both bifurcate from the non-trivial equilibrium $(0, 0)$, while the PW bifurcate in secondary bifurcations from the SW and TW branches (fig. 1). Analysis of eqs. (3a), (3b) also shows that if $AF < 0$ and $AF - CE > 0$ the PW can undergo a tertiary Hopf bifurcation to a three-frequency pulsing traveling wave (PW3). These waves lie on a three-torus and have one $O(2)$ frequency, one $O(\nu_1, \nu_2)$ frequency and a third frequency $d\phi/dt$ of order $A^2 z$. An example of such a wave, again constructed using the representation (4), (5), is shown in figs. 3a, 3b. As the bifurcation parameter increases the amplitude of the corresponding oscillation in $(\rho, z)$ increases, as does its period. The period becomes infinite when a heteroclinic connection forms joining the steady states, traveling waves and standing waves (fig. 1). The details of this process depend on fifth order terms in eqs. (3a), (3b).

In fig. 4 we show a nearly heteroclinic wave for the case $\nu_1 = -0.1, \nu_2 = 0.109, A = 1.0, C = 3.0, E = -3.0, 0.5$

![Fig. 4. A nearly heteroclinic wave, showing successive transitions from steady state behavior to a traveling wave and then a standing wave.](image-url)
$F = -1.0$, with eq. (3b) modified by the retention of the quintic term $-1.6z^5$. For these coefficients the heteroclinic connection forms at $\nu_2 = -\nu_1 + \nu_1^2 = 0.11$. Consequently the wave spends a considerable amount of time looking like each of the three states visited by the limit cycle, with rapid transitions between them. Note that since the decoupling of $\theta$ from the $p, z$ variables is a consequence of an $S^1$ normal form symmetry introduced by the normal form coordinate transformations, one expects to find in the original system transversal intersections of the stable and unstable manifolds, and hence the appearance of chaotic waves. This mechanism for generating chaotic waves is distinct from those studied by Knobloch et al. [7] and Knobloch and Moore [8]. Details of the relevant generic phenomena arising from breaking the $S^1$ symmetry can be found in ref. [9].

3. An example: magnetoconvection

We have identified the codimension-two bifurcation discussed above in a model of convection in a vertical magnetic field. This model is a Gelerkin truncation of the governing partial differential equations with stress-free boundary conditions at the top and bottom and periodic boundary conditions in the horizontal. Only those modes that contribute to the direction of branching of the various solutions from the trivial state are included. The model is thus minimal in the sense defined in ref. [5], and consists of the following equations for the eleven (real) modal amplitudes that must be retained,

$$
\begin{align*}
\dot{a} &= -\sigma a + \alpha b - \gamma q \{ d + \frac{1}{2} (\varpi - 1) (g - g^*) \} \\
&\quad + (3 - \varpi) d^* e + \frac{1}{2} (\varpi - 1) a (f - f^*) , \\
\dot{b} &= b + a - ac - \frac{1}{2} (f - f^*) , \\
\dot{c} &= -\varpi c + \frac{1}{2} \varpi (ab^* + a^* b) , \\
\dot{d} &= -\zeta d + a - a^* e + \frac{1}{2} a (g - g^*) + \frac{1}{2} d (f - f^*) , \\
\dot{e} &= -\zeta (4 - \varpi) e + \varpi ad , \\
\dot{f} &= -2\gamma \varpi g - 4\varpi d^* e , \\
\dot{g} &= 2f - \zeta \varpi g + \frac{1}{2} \varpi (ad^* - da^*) .
\end{align*}
$$

In these equations the amplitudes $a(t), b(t), d(t),$ $e(t)$ are complex, $c(t)$ is real, and $f(t), g(t)$ are imaginary. The constants $\sigma, \gamma$ and $\varpi$ are defined by

$$
\begin{align*}
\sigma &= \frac{k^2}{p^3} R , \\
\gamma &= \frac{\pi^2}{p^2} Q , \\
\varpi &= \frac{4\gamma^2}{p} ,
\end{align*}
$$

where $p = k^2 + \pi^2$ and $k$ is the horizontal wavenumber of the instability. Here $R$ is the Rayleigh number, $Q$ is the Chandrasekhar number, $\sigma$ is the Prandtl number and $\zeta$ denotes the ratio of ohmic to thermal diffusivity. These equations represent a generalization of the five-mode model studied by Knobloch et al. [10] to periodic boundary conditions in the horizontal. The equations have previously been used to calculate the coefficients in the Takens–Bogdanov normal form describing the interaction of small amplitude traveling waves, standing waves and steady states near the trivial state since they are constructed to agree with the partial differential equations to the required order [11].

We have located the codimension-two bifurcation in $(\sigma, \gamma, \varpi, \zeta, \psi)$ parameter space, and carried out the center manifold reduction of eqs. (6) at such points, determining the normal form coefficients $A, B, ..., E'$ in terms of the system parameters. We omit the details. In figs. 5a, 5b we show the standing and traveling waves for particular values of the system parameters. These solutions are stable. In figs. 5c, 5d we show a stable pulsing traveling wave. All of these solutions have been obtained by integrating eqs. (6) for parameter values suggested by the local analysis near the codimension-two point.

4. Conclusions

In this paper we have shown that a relatively simple codimension-two bifurcation from a circle of nontrivial equilibria in a system with $O(2)$ symmetry can give rise not only to traveling waves of considerable complexity, but to chaotic waves as well. This behavior is generic in two-parameter families of vector fields with $O(2)$ symmetry. Consequently we expect the behavior described above to be present in a variety of physical systems, and in particular in ones described by partial differential equations with periodic boundary conditions. Perhaps the simplest possibility is provided by two-dimensional Boussi-
Fig. 5. Solutions obtained by numerical integration of eqs. (6). (a) A standing wave (SW) obtained for $r=1.2986$, $q=0.27144$, $\sigma=1.0$, $\zeta=0.0475$, $\sigma=0.31005$. (b) A traveling wave (TW) for $r=6.75215$, $q=5.32112$, $\sigma=\frac{1}{3}$, $\zeta=\frac{1}{3}$, $\sigma=0.135304$. (c) A pulsing traveling wave (PW) for $r=6.72498$, $q=5.34981$, $\sigma=\frac{1}{3}$, $\zeta=\frac{1}{3}$, $\sigma=0.135304$. (d) The phase $\phi(t)$ of the PW defined by $a(t)=|a(t)|e^{i\phi}$, where $a(t)$ is given by (6a). The mean slope is associated with the constant wave speed of the TW from which the PW is born. The oscillations about this slope are responsible for the pulsing of the wave.
nesq convection in a pure fluid. Here it is known that at sufficiently supercritical Rayleigh numbers a reflection-preserving Hopf bifurcation does occur on the branch of nontrivial equilibria [12], with a reflection-breaking steady state bifurcation likely to be present as well [13]. Other systems expected to behave similarly are compressible convection and compressible magnetoconvection. The bifurcation discussed here is also relevant to the unfolding of the degeneracy $D/M = 1$ in the Takens–Bogdanov bifur-
cation with $O(2)$ symmetry [4]. Finally, it also occurs as a degeneracy in the normal form for the interaction of two steady state bifurcations with $O(2)$ symmetry and wavenumbers $mk$ and $nk$, both when $m, n (m<n)$ are two relatively prime integers [14] or when $m=n$ [15].

Acknowledgement

We have benefitted from discussions with J.M. Massaguer and A. Rucklidge.

References