Spatially Localized Magnetoconvection

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Abstract. Numerical continuation is used to compute branches of time-independent, spatially localized convectons in an imposed vertical magnetic field focusing on values of the Chandrasekhar number $Q$ in the range $10 < Q < 10^3$. The calculations reveal that convectons initially grow by nucleating additional cells on either side but with the build-up of field outside owing to flux expulsion the convectons are able to transport more heat only by expanding the constituent cells. Thus at large $Q$ and large Rayleigh numbers convectons consist of a small number of broad cells.

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1. Introduction

Convections are spatially localized regions of convection embedded in a quiescent background. States of this type were found in numerical studies of magnetocovection, i.e., convection in an imposed vertical magnetic field, in both two (Blanchflower 1999) and three spatial dimensions (Blanchflower and Weiss 2002, Houghton and Bushby 2011), although there is evidence for similar states in earlier work on two-dimensional natural convection in a vertical slot (Ghorayeb and Mojtabi 1997). The existence and stability of these structures has elicited considerable interest in recent years (Knobloch 2008). In a recent paper Lo Jacono et al. (2011) used numerical continuation to compute branches of two-dimensional magnetohydrodynamic convections in a periodic domain with stress-free, fixed temperature boundary conditions at the top and bottom of a horizontal layer of a conducting Boussinesq fluid, focusing on the case $Q = 4$, where the Chandrasekhar number $Q$ provides a dimensionless measure of the strength of the imposed magnetic field. This relatively small value was selected since in this case no overstability is present for the values of the magnetic Prandtl number ($\zeta = 0.1$, and $\zeta = 0.5$) used in the computations. These values of $\zeta$ bracket the transition from subcritical to supercritical periodic convection, and allowed us to study the changes in the properties of magnetohydrodynamic convections as the interval of coexistence between the conduction state and spatially periodic convection shrinks and eventually disappears. In contrast to other convection-bearing systems Lo Jacono et al. (2011) found that the convections persisted into the supercritical regime and, following Cox and Matthews (2001), attributed this fact to the presence of a conserved quantity, the imposed magnetic flux.

In this paper we use the same procedure to extend the results of Lo Jacono et al. (2011) to larger values of $Q$. Specifically we report here on the results obtained for (i) $Q = 100$, (ii) $Q = 400$ and (iii) $Q = 1000$, all with $\zeta = 0.5$ and $\sigma = 1$, where $\sigma$ denotes the Prandtl number. For this parameter choice the width of the diffusive boundary layers, measured by the Péclet and magnetic Reynolds numbers $Pe \equiv Uh/\kappa$, $Re_m \equiv Uh/\eta$, remains of order $h$, the layer depth, provided the imposed Rayleigh number $Ra$ remains of order $Ra_c$, the critical Rayleigh number for the onset of convection. Here $\kappa$ and $\eta$ are the thermal and ohmic diffusivities and $U$ denotes a characteristic convective speed. However, with increasing values of $Ra$ the boundary layers become thinner and thinner leading to a substantial increase in the resolution requirements of our spectral element continuation code (Assemat et al. 2008, Bergeon and Knobloch 2008). All our computations are carried out in a periodic domain with aspect ratios $\Gamma = 10\lambda_c$ or $\Gamma = 20\lambda_c$, where $\lambda_c$ is the convection wavelength at $Ra = Ra_c$.

The larger values of $Q$ considered here are responsible for two important differences from the $Q = 4$ case considered by Lo Jacono et al. (2011). Specifically the onset of steady convection is delayed to larger values of $Ra$ but because the wavelength $\lambda_c$ is a decreasing function of $Q$ the bifurcation to steady convection remains supercritical (see Eq. (10) below). In addition, the onset of steady convection is preceded by a number of
Hopf bifurcations to modes with other horizontal wavenumbers. We do not compute the resulting branches of standing and traveling waves, and refer to Weiss (1981), Knobloch and Proctor (1981), Knobloch et al. (1981), and Dangelmayr and Knobloch (1986) for further discussion. Owing to the cost involved we do not continue the eigenvalues describing the stability of the convectons, but expect, following Blanchflower (1999), that the solutions we compute will be stable in the vicinity of the minimum Rayleigh number for the existence of convectons. As discussed further below this Rayleigh number is a strong function of the length of the domain used for the computations. This is in turn a consequence of the conservation of magnetic flux across the layer.

Convection in an imposed magnetic field is described by the dimensionless equations

\[ u_t + (u \cdot \nabla) u = -\nabla P + \sigma RaT \hat{z} + \sigma \zeta Q (\nabla \times B) \times B + \sigma \nabla^2 u, \]
\[ T_t + (u \cdot \nabla) T = \nabla^2 T, \]
\[ B_t + (u \cdot \nabla) B = (B \cdot \nabla) u + \zeta \nabla^2 B, \]

(1, 2, 3)

together with the incompressibility condition
\[ \nabla \cdot u = 0. \]

(4)

Here \( u \) is the velocity, \( P \) is the pressure, \( T \) is the temperature and \( B \) is the magnetic field, and the equations have been nondimensionalized with respect to the thermal diffusion time \( h^2/\kappa \) in the vertical. The system is characterized by four dimensionless parameters, the Rayleigh number \( Ra \equiv \alpha \Delta T h^3/\kappa \nu \) measuring the strength of the thermal forcing (i.e., temperature difference \( \Delta T \)), the Chandrasekhar number \( Q \equiv B_0^2 h^2/\mu_0 \rho \eta \nu \) measuring the strength of the ambient magnetic field \( B_0 \), and the two Prandtl numbers \( \sigma = \nu/\kappa \) and \( \zeta = \eta/\kappa \). Here \( \nu \) is the kinematic viscosity.

In two dimensions it is convenient to rewrite these equations in terms of the temperature departure \( \theta \) from the conduction profile, \( T = 1 - z + \theta \), and the flow-induced magnetic field perturbation \( \mathbf{b} \) from the imposed vertical field, \( \mathbf{B} = \hat{z} + \mathbf{b} \). With \( u \equiv \nabla \times \psi \hat{y} \) and \( \mathbf{b} \equiv \nabla \times A \hat{y} \) these equations take the form

\[ \nabla^2 \psi_t + J(\psi, \nabla^2 \psi) = \sigma Ra \theta_x + \sigma \zeta Q J(x + A, \nabla^2 A) + \sigma \nabla^4 \psi, \]
\[ \theta_t + J(\psi, \theta) = \psi_x + \nabla^2 \theta, \]
\[ A_t + J(\psi, A) = \psi_z + \zeta \nabla^2 A. \]

(5, 6, 7)

We study these equations with stress-free fixed temperature boundary conditions
\[ \psi = \psi_{zz} = \theta = 0 \quad \text{on} \quad z = 0, 1 \]

(8)

and for the magnetic field we take the force-free boundary conditions
\[ A_z = 0 \quad \text{on} \quad z = 0, 1. \]

(9)

In addition we impose periodic boundary conditions in the horizontal with period \( \Gamma \). With these boundary conditions the quantity \( \bar{A} \equiv \Gamma^{-1} \int_0^\Gamma \int_0^1 A(x, z, t) \, dx \, dz \) remains constant in time.

These equations have a trivial homogeneous solution \( (\psi, \theta, A) = (0, 0, 0) \) corresponding to pure thermal conduction. Linear stability theory shows that this state
loses stability with respect to exponentially growing disturbances with wavenumber $k$ when $Ra$ reaches $Ra(k) = (p^3/k^2)[1 + (\pi^2/p^2)Q]$, where $p \equiv \pi^2 + k^2$. The critical Rayleigh number $Ra_c \equiv Ra(k_c)$ and the wavenumber $k_c$ of the critical disturbance, are obtained by minimizing the onset Rayleigh number with respect to $k$. Neither depends on the parameters $\zeta$ and $\sigma$. Classical theory shows that at $Ra = Ra_c$ a steady state bifurcation creates a branch of spatially periodic states with wavenumber $k_c$ (equivalently, wavelength $\lambda_c$) that bifurcates subcritically when

$$1 + \frac{\pi^2}{p^2} \left[ 1 + \frac{2\pi^2(p-2\pi^2)}{p(p-\pi^2)} \frac{1}{\zeta^2} \right] Q < 0$$

and supercritically otherwise (Weiss 1981, Knobloch et al. 1981). Moreover, when condition (10) holds and the problem is formulated on the real line this instability gives rise, in addition, to two families of subcritical branches of spatially localized states of opposite parity under reflection in a vertical plane through their midpoint. Each solution type can, in addition, be reflected in the layer midplane and/or translated horizontally to generate other members of each family (Lo Jacono et al. 2010, Mercader et al. 2011).

We mention that the conduction state loses stability with respect to oscillations at $Ra(k) = \sigma^{-1}(p^3/k^2)(\sigma + \zeta)[(1 + \zeta) + \sigma\zeta(\pi^2/p^2)Q/(1+\sigma)]$, a quantity that is minimized by taking $k = k_H$. The corresponding Rayleigh number and oscillation frequency will be referred to as $Ra_H$, $\omega_H$, respectively, where $\omega_H^2 = (p_H/k_H^2)[-\zeta^2 + (1-\zeta)\sigma\zeta(\pi^2/p_H^2)Q/(1+\sigma)]$.

2. Results

2.1. $Q = 100$

For $Q = 100$ linear theory in an infinite domain yields $Ra_c \approx 2653.7$, $k_c \approx 3.7015$, independently of the value of $\zeta$, while the first Hopf bifurcation from the conduction state occurs at $Ra = Ra_H \approx 2429.2$, with critical wavenumber $k_H \approx 2.7561$ and frequency $\omega_H \approx 6.8636$. Since $\Gamma = 10\lambda_c$, where $\lambda_c \equiv 2\pi/k_c$, the steady state instability sets in at $Ra = Ra_c$ and gives rise to a (supercritical) spatially periodic state we call $P_{10}$, consisting of 20 counter-rotating rolls per period $\Gamma$. The steady, spatially localized, states of interest here bifurcate from this state in a secondary bifurcation that takes place already at small amplitude.

Figure 1 shows the kinetic energy $E \equiv (1/2) \int_0^\Gamma \int_0^1 |u|^2 \, dx \, dz$ as a function of the Rayleigh number $Ra$ for the periodic state $P_{10}$, together with a pair of spatially localized states of even ($L^+$, dashed line) and odd ($L^-$, solid line) parity. This representation does not distinguish between states related by reflection symmetry in the midplane or by translations. The $P_{10}$ solutions undergo a secondary bifurcation already at small amplitude through an instability first identified by Matthews and Cox (2000). This bifurcation is strongly subcritical and generates two pairs of branches of opposite parity spatially localized structures (Fig. 1(b)), exactly as in the case $Q = 4$ (Lo Jacono et al. 2011). These branches are initially exponentially close and the corresponding structures...
are weakly localized and unstable. However, with increasing amplitude (decreasing \( Ra \)) the structures rapidly localize before turning around in saddle-node bifurcations and beginning to grow in length by nucleating new convection cells at either end. In the following we denote the leftmost saddle-node on each branch by \( Ra = Ra^{*}_{\text{odd/even}} \).

During this growth process the midplane magnetic potential \( A(x, z = 1/2) \) develops a nonzero slope which remains almost constant in the center of the structure despite the changing Rayleigh number and the growing number of rolls on either side. The rolls imprint a staircase structure on this overall slope, by an essentially kinematic flux expulsion process elucidated by Weiss (1966), and the overall appearance of \( A(x, z = 1/2) \) is almost identical for both even and odd states. This is a consequence of the fact that the cells expel magnetic field regardless of their direction of rotation. Thus a churning array of cells expels field to the boundary of the array regardless of whether the outermost cells rotate in the same or opposite directions. The mean slope of \( A(x, 1/2) \) represents the reduction (negative slope) in the vertical magnetic field within the structure and enhancement (positive slope) outside (Figs. 2 and 3). Thus the resulting vertical magnetic field has an essentially piecewise constant profile: reduced within the structure, except at cell boundaries, and enhanced outside.

The growth process manifests itself in a series of loops in \( \mathcal{E} \) as \( R \) increases (Fig. 1). These loops are associated with the nucleation of additional cells as the number of cells within the convection grows (Fig. 2). The origin of the loops themselves can be traced to
a gradual increase in the wavelength of the cells within the convecton as $Ra$ increases – such an increase is possible because the convectons are localized – followed by a decrease in wavelength once new cells nucleate at a saddle-node on the right (Figs. 2 and 3). It is as if the new cells, one on each side, squeeze the convecton between them. This effect grows as the new cells grow to their saturated amplitude, a process that takes place along the lower portion of each loop. Beyond the saddle-node on the left all the cells within the convecton are of the same strength and the wavelength starts to increase again. Thus despite the increase in the number of cells the convectons along the lower portion of each loop are shorter and hence occupy a smaller fraction of the domain than prior to the nucleation event, resulting in decreased kinetic energy $E$ (Figs. 2 and 3). This process repeats as $Ra$ increases with the loops getting larger as the associated changes in energy become larger, before the domain is full. Once not enough space is available to insert new cells on either side the structure ceases to grow, and only the vigor of the convecting cells increases with further increase in $Ra$ (Fig. 1(a)). The resulting states resemble the defect states studied elsewhere (Bergeon and Knobloch 2008, Bergeon et al. 2008, Mercader et al. 2009) and the resulting structure resembles slanted snaking (Firth et al. 2007, Dawes 2008).
2.2. $Q = 400$

With increasing $Q$, the number of loops gradually decreases. For example, when $\Gamma = 10\lambda_c$ and $Q = 250$ (not shown) only two loops are present on each convecton branch, while only one is present when $Q = 400$ (Fig. 4). This fact implies a dramatic change in the convecton growth process since the presence of loops indicates nucleation of new cells. Thus the convecton must manage to transport more and more heat as the Rayleigh number increases without adding new cells. To see how this is possible we now present the results for $Q = 400$. For this value of $Q$, linear theory yields $Ra_c \approx 7185.9$, $k_c \approx 4.7942$, independently of the value of $\zeta$, while the first Hopf bifurcation from the conduction state occurs already at $Ra = Ra_H \approx 4658.9$, with critical wavenumber $k_H \approx 3.4395$ (for $\zeta = 0.5$, $\sigma = 1$).

Figures 5 and 6 show that for $Q = 400$ the vertical magnetic field, $B(x, 1/2)$, has a more nonlinear, square-wave profile in the saturated state. Indeed, the profiles are no longer monotonic and the slight dips on either side of an upflow or a downflow are the result of advection of the field by the rising or falling plume. A careful examination of the solution profiles along the remaining loop reveals that the wavelength is no longer selected by the value of $Ra$ alone. Instead Figs. 5 and 6 reveal a significant dependence on the amplitude of state, with the low energy state having a substantially shorter
Figure 4. Bifurcation diagram for $Q = 400$ in a (a) $\Gamma = 10\lambda_c$, (b) $\Gamma = 20\lambda_c$ domain showing the energy $E$ as a function of the Rayleigh number $Ra$ for the periodic branch $P_{10}$ and the branches of even ($L^+$, dashed line) and odd ($L^-$, solid line) parity convections.

wavelength than the large amplitude states even at comparable Rayleigh numbers. The figures also reveal a considerable dependence of the flux expulsion process on the amplitude of the localized structures, as measured by the slope of $A(x, z = 1/2)$ outside the structure (dashed profiles), with fully developed convections much more efficient at flux expulsion. This is primarily a consequence of a larger magnetic Reynolds number $Re_m$. The increased magnetic field outside the convection in turn suppresses the nucleation of new cells and leads to a new growth process as $Ra$ increases. Thus flux conservation leads to suppression of nucleation. For $Q = 400$ this occurs after the even convection has grown to four cells and the odd convection to three cells (Figs. 5 and 6) as expected from the presence of a single loop on each branch. Instead one finds that beyond the loop the width of the individual cells in each of these states begins to increase monotonically with increasing $Ra$ until the domain is almost full. This new growth process thus results in a cellular state with a relatively large wavelength (Fig. 7), with defects or holes separated by distance $\Gamma$.

To confirm this picture we have recomputed our $Q = 400$ results on a domain with $\Gamma = 20\lambda_c$. Figure 4(b) shows the presence of two loops on the branch of even convections and three loops on the branch of odd convections instead of one loop on each branch, and indeed the convections now consist of six cells (even) or seven cells (odd) when the loops cease and the existing cells start to stretch (not shown). These results confirm that the transition from nucleation of new cells to the stretching of existing cells is triggered by
Figure 5. (a) Bifurcation diagram for $Q = 400$ in a $\Gamma = 10\lambda_c$ domain showing details of the loop structure in Fig. 4(a) for even (solid line) and odd (dashed line) convections. (b) Even parity convections at the locations indicated in (a). Top panels: contours of the streamfunction $\psi(x, z)$. Bottom panels: the magnetic potential $A(x, 1/2)$ (dashed profile) and the corresponding vertical magnetic field $B(x, 1/2) \equiv 1 + A_x(x, 1/2)$ (solid profile), both at midplane $z = 1/2$.

a critical strength of the ambient magnetic field.

In Fig. 7 we compare the large Rayleigh number $Q = 400, \Gamma = 10\lambda_c$ solutions with the corresponding solutions for $Q = 100, \Gamma = 10\lambda_c$. We see that the wavelength of the structure when $Q = 400$ is much larger than when $Q = 100$ but that it remains uniform even at $Ra \sim 9000$. At these large values of $Ra$ the solution branches increase monotonically, much as in Fig. 4(a), with no additional features.

2.3. $Q = 1000$

For $Q = 1000$ linear theory yields $Ra_c \approx 15207, k_c \approx 5.6842$, independently of the value of $\zeta$, while the first Hopf bifurcation from the conduction state occurs already at $Ra = Ra_H \approx 8415.3$, with critical wavenumber $k_H \approx 4.0691$ (for $\zeta = 0.5, \sigma = 1$).

For this large value of $Q$ the behavior of the convecton branches $L_{10}^\pm$ and $L_{20}^\pm$ is considerably different (Fig. 8). The regular looping seen at smaller values of $Q$ is now absent and there are substantial differences in the behavior of the odd and even convections. The results are also sensitive to the domain length $\Gamma$ with the leftmost saddle-nodes shifting towards smaller Rayleigh numbers as the domain increases. This is again a consequence of flux conservation. On smaller domains flux conservation increases the magnetic field strength outside the convecton thereby limiting convection
Figure 6. (a) Bifurcation diagram for $Q = 400$ in a $\Gamma = 10\lambda_c$ domain showing details of the loop structure in Fig. 4(a) for odd (solid line) and even (dashed line) convectons. (b) Odd parity convectons at the locations indicated in (a). Top panels: contours of the streamfunction $\psi(x, z)$. Bottom panels: the magnetic potential $A(x, 1/2)$ (dashed profile) and the corresponding vertical magnetic field $B(x, 1/2) \equiv 1 + A_x(x, 1/2)$ (solid profile), both at midplane $z = 1/2$.

(a) $Ra = 9023.85$
(b) $Ra = 9035.38$
(c) $Ra = 9048.70$
(d) $Ra = 9035.71$

Figure 7. Profiles for even and odd convectons for (a-b) $Q = 100$, $\Gamma = 10\lambda_c$, and (c-d) $Q = 400$, $\Gamma = 10\lambda_c$ at the Rayleigh numbers indicated, showing large wavelength nearly periodic structures filling the domain at large $Ra$ when $Q$ is also large.

to larger Rayleigh numbers than in larger domains. This effect is much larger for odd convectons since the first saddle-node on the left corresponds to a single convection cell embedded in a conduction background.

Figure 9 shows sample solutions at the left saddle-nodes along (a) $L_{20}^+$ and (b) $L_{20}^-$ in a $\Gamma = 20\lambda_c$ domain. As shown in Fig. 8(b) the $L_{20}^+$ branch goes through a loop at small amplitude after which the solution consists of four cells. However, near the spike that follows (this spike is clearly visible in Fig. 8(b)) the even parity solution briefly splits into a bound state of two oppositely-oriented single cell convections before recombining into a four cell state in which the outer cells retain their dominance. With increasing amplitude (Fig. 8(c)) the inner cells grow in both amplitude and wavelength, forming,
Figure 8. (a) Bifurcation diagram for $Q = 1000$ in a $\Gamma = 10\lambda_c$ domain (dashed curves) and a $\Gamma = 20\lambda_c$ domain (solid curves) showing the energy $\mathcal{E}$ as a function of the Rayleigh number $Ra$ for the periodic branches $P_{10}$ and $P_{20}$ and the branches of even and odd parity convections $L_{10}^\pm$ and $L_{20}^\pm$. (b) Enlargement of (a) near the primary bifurcation. (c) Enlargement of (a) on an intermediate scale, below the leftmost saddle-nodes on the $L_{10}^+$ and $L_{20}^+$ branches.
Figure 9. Solutions when $Q = 1000$, $\Gamma = 20\lambda_c$. (a) Streamfunction at left saddle-nodes along the $L_{20}^+$ branch ($Ra \approx 11880, 7906, 8360, 6701, 6717$) and at large $Ra$ ($Ra = 12187$). (b) Streamfunction at the left saddle-node along the $L_{20}^-$ branch ($Ra \approx 5612$), and at large $Ra$ ($Ra = 15039$). In each case the kinetic energy $E$ increases from bottom to top.

by the time the leftmost saddle-node is reached, a state consisting of four similar cells. At this point the cells start to stretch resulting in the four cell state shown in Fig. 9(a) at $Ra = 12187$. In contrast to the even parity states the energy of the odd parity states $L_{20}^-$ grows much more slowly (Fig. 9(b)). These states undergo only a single saddle-node at which the solution takes the form of a single convecting cell. With increasing amplitude the branch $L_{20}^-$ turns around and the solution gradually adds narrow convection cells on either side, without exhibiting either additional loops or additional saddle-nodes. This behavior, already seen when $Q = 4$ (Lo Jacono et al. 2011), resembles the “smooth snaking” first described in the context of a nonlocal oscillon model by Dawes and Lilley (2010). The resulting solution, shown in Fig. 9(b) at $Ra = 15039$, resembles the narrow convection cells traditionally expected of convection in a strong magnetic field (Julien et al. 1999; Matthews 1999) but differs dramatically from the corresponding large amplitude even state in Fig. 9(a).

2.4. Location of the leftmost saddle-nodes

In Fig. 10 we show the location of the leftmost saddle-nodes, $Ra = Ra^*_\text{odd/even}$, on the branches of odd and even convections for different values of $Q$ in the range $100 \leq Q \leq 1000$ computed for $\Gamma = 20\lambda_c$. The states corresponding to these saddle-nodes are well-localized convections consisting of one or two cells only. In contrast to the suggestion of Dawes (2007) based on results in domains with $\Gamma = 6$ and $\Gamma = 10$, i.e., for substantially smaller domains than in the present work, we do not see a $\Gamma$-independent power law dependence on $Q$. Our computations suggest (Fig. 8) that the location of the leftmost saddle-nodes is in fact a sensitive function of the aspect ratio $\Gamma$ even for the relatively large domains employed in our computations. This is a consequence of flux conservation which plays a fundamental role in finite domains even for nominally localized structures (Lo Jacono et al. 2011). Despite this difference we expect to find stable convections above and close to the saddle-node lines shown in the figure, as found by Blanchflower (1999) and confirmed by Dawes (2007) using direct numerical simulations.
Figure 10. The location in a log-log plot of the leftmost saddle-nodes $Ra = Ra_{\text{even}}^*$ for even convectons (open circles) and $Ra = Ra_{\text{odd}}^*$ for odd convectons (solid squares) as functions of $Q$, computed with $\Gamma = 20\lambda_c$.

3. Discussion

In this paper we have extended the results obtained by Lo Jacono et al. (2011) for magnetoconvection with $Q = 4$ to substantially larger values of $Q$. In order to reduce resolution requirements we used the magnetic Prandtl number $\zeta = 0.5$ and considered periodic domains $\Gamma = 10\lambda_c$ and $\Gamma = 20\lambda_c$ with 10 and 20 critical wavelengths, respectively. For these parameter values the periodic state bifurcates supercritically. Despite this spatially localized states of odd and even parity are present for Rayleigh numbers below $Ra = Ra_c$, indeed substantially below $Ra = Ra_{c}^*$. Although we have presented detailed results only for $Q = 100$, 400 and 1000 we have performed similar computations for a range of values of $Q$ in the range $10 \leq Q \leq 1000$. These suggest that there are two qualitatively distinct types of behavior. For $10 \leq Q \leq 100$ the system behaves much as shown in Figs. 1–3 with orderly growth of the localized structure that manifests itself in the presence of loops in the energy $E$. In this regime the effects of flux conservation are limited to the regime in which the convectons almost fill the entire domain (Fig. 7(a,b)). For larger $Q$ these loops gradually disappear as the build-up of the magnetic field outside the structure suppresses the nucleation of new cells on either side. We conjectured that in larger domains the suppression of nucleation will be delayed until the ambient magnetic field reaches the strengths at which nucleation is suppressed in domains with aspect ratio $\Gamma = 10\lambda_c$. This conjecture is confirmed in Fig. 4(b) which reveals the presence of three loops when $\Gamma = 20\lambda_c$, all other parameters remaining the same.

The appearance of broader cells within the convectons as $Q$ increases shows that extrapolation of linear theory predictions to strongly nonlinear solutions is dangerous
– linear theory predicts that the wavelength of convection cells varies with $Q$ as $Q^{-1/6}$ and hence that cells become narrower as the magnetic field strength increases. For the largest magnetic field strengths examined, $Q = 1000$, we found narrow cell convectons at large Rayleigh numbers to be of odd parity only, with the corresponding even parity states consisting of a small number of broad cells. It is possible, but remains unproven, that for yet larger values of $Q$ these broad cell states undergo a transition to narrow cells with a concomittant decrease in slope of $E(Ra)$ owing to increased viscous dissipation.

Based on earlier results with $Q = 4$ we conjecture that similar results to those reported here will obtain for smaller values of the diffusivity ratio $\zeta$. For $\zeta \ll 1$ the bifurcation to periodic convection becomes strongly subcritical but since the convectons reported here are already strongly subcritical we anticipate little qualitative effect on these states. However, since the magnetic boundary layers become thinner and thinner as $\zeta$ decreases we expect flux expulsion to be more complete at any given Rayleigh number, and hence expect the leftmost saddle-nodes to shift to yet smaller values of $Ra$.

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**References**


