Complex dynamics in the 1:3 spatial resonance

J. Porter, E. Knobloch*

Department of Physics, University of California, Berkeley, CA 94720, USA

In memory of John David Crawford

Abstract

The interaction between two steady-state bifurcations with spatial wave numbers in the ratio 1:3 is considered. Periodic boundary conditions are assumed. The resulting O(2) equivariant normal form, truncated at third order, exhibits a number of global bifurcations that may result in complex dynamics. The origin of this behavior is elucidated with the help of careful numerical simulations and analysis of appropriate return maps. The results generalize to other 1:n resonances. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Spatial resonance; Global bifurcations; Complex dynamics

1. Introduction

The analysis of wave–wave interactions plays a vital role in understanding the behavior of continuous systems [1]. In Hamiltonian systems such interactions are responsible for mode conversion as well as various types of instabilities. In driven dissipative systems related processes lead to wave number selection [2]. Even in systems, such as convection, that do not support wave-like disturbances, mode–mode interactions are often responsible for the presence of unexpected dynamics. The basic mechanism is described most clearly by Dangelmayr [3] in the context of the interaction between two spatial modes with wave numbers in the ratio \( m:n, m < n \), assumed to be coprime. For problems on the real line with periodic boundary conditions and symmetry under spatial reflection \( (x \rightarrow -x) \) the resulting problem is equivariant under the standard action on \( C^2 \) of the group O(2) of rotations and reflections of a circle. If \( m > 1 \) pure modes \( P_m, P_n \) bifurcate from the trivial state followed by secondary bifurcations to mixed modes which in turn can undergo either a Hopf bifurcation leading to standing oscillations, or a parity-breaking bifurcation leading to drifting mixed modes, i.e., to traveling waves. The analysis also applies to problems with Neumann (and in appropriate cases, Dirichlet) boundary conditions since such problems have hidden O(2) symmetry [4,5], but the traveling waves are then absent. On the real line the competing wave numbers may be close to one another and the wave number selection process is then described by the Ginzburg–Landau equation, and is the consequence of the so-called Eckhaus or sideband instability. As discussed

* Corresponding author. Fax: +1-510-643-8497.
E-mail address: knobloch@physics.berkeley.edu (E. Knobloch)
by Tuckerman and Barkley [6], on a large but finite domain the Ginzburg–Landau equation gives rise to a number of mode interactions between adjacent wave numbers that are responsible for the presence of multiple stable steady states at finite amplitude, familiar from studies of finite domains [7,8]. However, as shown by Mizushima and Fujimura [9] the familiar Eckhaus boundary for convection rolls is substantially modified by the presence of strong resonances, particularly those involving the wave number 1. Of these the 1:2 steady-state interaction has been extensively studied and is known, under appropriate circumstances, to lead to attracting structurally stable heteroclinic cycles [10,11]. Mizushima and Fujimura pointed out, however, that in Rayleigh–Bénard convection with identical boundary conditions at the top and bottom of the layer (and no non-Boussinesq effects) it is the 1:3 resonance that plays an important role. This is because the resulting midplane reflection symmetry pushes the resonant terms for the 1:2 resonance to fifth order [12], while the resonant terms for the 1:3 resonance remain at third order. Prat et al. [13] show by direct integration of the two-dimensional equations describing Rayleigh–Bénard convection with periodic boundary conditions that the 1:3 resonance does indeed organize the wave number selection process in large regions of parameter space, despite the fact that it is always shielded by instabilities to other modes. We use these results to motivate our interest in the 1:3 resonance. Related issues arise in the theory of unstable electrostatic waves when the equilibrium electron velocity distribution function is reflection-symmetric [14].

Explicit computation of the normal form coefficients for the 1:3 resonance in Rayleigh–Bénard convection [9] indicates that interesting time-dependence may be present for sufficiently low Prandtl numbers. Although this is so even with Neumann boundary conditions [8], with periodic boundary conditions Mizushima and Fujimura noted the presence of both periodic and quasi-periodic traveling waves. We show below that the solutions of the equations describing the 1:3 mode interaction are in fact much richer. In particular we describe, via careful numerical studies, the fate of the quasi-periodic oscillations, both in the case studied by Mizushima and Fujimura, and in others. We uncover a number of new global bifurcations involving circles of steady states and of standing waves, as well as the trivial state. Continuous deformation from one heteroclinic cycle to another is found to explain a number of features of the bifurcation diagrams determined numerically. This explanation is supported by analytically constructed return maps which yield, as a byproduct, conditions on the eigenvalues and Floquet multipliers of the various solution types for the presence of stable complex dynamics. In contrast to the behavior identified in [10,11] for the 1:2 resonance much of the behavior of interest in the present problem is associated with structurally unstable heteroclinic cycles, although structurally stable heteroclinic cycles can also be present.

This paper is organized as follows. In Section 2 we construct the truncated normal form equations studied in this paper, and rewrite these in several equivalent but useful forms. In Sections 3 and 4 we summarize the properties of the simplest solutions and of the bifurcations leading to them. Section 5 looks at the geometry of the various representations of the vector field. Section 6 summarizes our numerical results for five choices of the normal form coefficients, chosen to produce nontrivial dynamics. In Section 7 we describe our interpretation of these results, and support it with a detailed discussion of the properties of the various return maps associated with the presence of the global connections identified in Section 6. These results are extended to other 1:n resonances in Section 8 and the results are summarized in Section 9.

2. The amplitude equations

We consider a two-parameter family of vector fields on $\mathbb{C}^2$ that are equivariant under the following representation of the symmetry group $O(2)$:

$$ T_\phi : (z_1, z_3) \mapsto (e^{i\phi} z_1, e^{3i\phi} z_3), \quad R : (z_1, z_3) \mapsto (\overline{z}_1, \overline{z}_3), $$

(1)
resulting from the effect of horizontal translations $T_\phi : x \rightarrow x + \ell, \phi \equiv \ell/k$, and reflection $R : x \rightarrow -x$ on scalar fields of the form
\[
\psi(x, t) = \Re[z_1(t) e^{ikx} + z_3(t) e^{3ikx}] + \cdots.
\] (2)

Here \( \cdots \) indicates higher harmonics whose amplitudes can be expressed in terms of the amplitudes $z_1, z_3$ of the primary modes. The equivariance requirement implies that $\dot{z}_j = f_j(z_1, z_3),\ j = 1, 3$, where
\[
T_\phi : f_j(z_1, z_3) = e^{-i\phi} f_j(e^{i\phi} z_1, e^{i3\phi} z_3), \quad R : f_j(z_1, z_3) = \bar{f}_j(\bar{z}_1, \bar{z}_3).
\] (3)

The symmetries (1) have three fundamental invariants [3,15], given by
\[
u = |z_1|^2, \quad v = |z_3|^2, \quad w = z_1^3 z_3 + z_3^3 z_1,
\] (4)

while the equivariants are generated by $(z_1, z_3)$ and $(\bar{z}_1^2 z_3, z_3^2)$. It follows that the most general equations for $(z_1, z_3)$ take the form
\[
\dot{z}_1 = p_1(u, v, w)z_1 + q_1(u, v, w)z_1^2 z_3, \quad \dot{z}_3 = p_3(u, v, w)z_3 + q_3(u, v, w)z_1^3,
\] (5)

where $p_j$ and $q_j$ are real-valued functions of the invariants $u, v, w$. To third order in the amplitudes, one therefore obtains the equations
\[
\dot{z}_1 = (\mu_1 + b_{11}|z_1|^2 + b_{13}|z_3|^2)z_1 + c_1 z_1^2 z_3, \quad \dot{z}_3 = (\mu_3 + b_{31}|z_1|^2 + b_{33}|z_3|^2)z_3 + c_3 z_3^3.
\] (6)

where $\mu_1$ and $\mu_3$ play the role of the unfolding parameters, and $b_{11}, b_{13}, b_{31}, b_{33}, c_1$ and $c_3$ are real, $\mu_j$-independent coefficients. These coefficients can be computed via center manifold (or Lyapunov–Schmidt) reduction of the governing equations at the 1:3 resonance, i.e., at the codimension-2 point $(\mu_1, \mu_3) = (0, 0)$, and in the following will be assumed to satisfy appropriate nondegeneracy conditions. Note that in contrast to the 1:2 resonance [10] no quadratic terms are present.

Eqs. (6) can be simplified by rescaling the amplitudes
\[
(z_1, z_3) \mapsto (|c_1 c_3|^{-1/4} z_1, |c_1 c_3|^{1/4} c_1^{-1} z_3)
\] (7)

to obtain the equivalent system
\[
\dot{z}_1 = (\mu_1 + d_{11}|z_1|^2 + d_{13}|z_3|^2)z_1 + z_1^2 z_3, \quad \dot{z}_3 = (\mu_3 + d_{31}|z_1|^2 + d_{33}|z_3|^2)z_3 + \sigma z_3^3,
\] (8)

where
\[
\begin{align*}
d_{11} &= |c_1 c_3|^{-1/2} b_{11}, & d_{13} &= |c_3|^{1/2} |c_1|^{-3/2} b_{13}, & d_{31} &= |c_1 c_3|^{-1/2} b_{31}, & d_{33} &= |c_3|^{1/3} |c_1|^{-3/2} b_{33}, & \sigma &= \text{sign}(c_1 c_3).
\end{align*}
\] (9)

It is these equations that are analyzed in the remainder of this paper. For some aspects of the discussion it will be helpful to rewrite these equations in terms of the real variables $z_j = x_j + i y_j$:
\[
\begin{align*}
\dot{x}_1 &= (\mu_1 + d_{11}|z_1|^2 + d_{13}|z_3|^2)x_1 + (x_1^2 - y_1^2) x_3 + 2 x_1 y_1 y_3, \\
\dot{y}_1 &= (\mu_1 + d_{11}|z_1|^2 + d_{13}|z_3|^2)y_1 + (x_1^2 - y_1^2) y_3 - 2 x_1 y_1 x_3, \\
\dot{x}_3 &= (\mu_3 + d_{31}|z_1|^2 + d_{33}|z_3|^2)x_3 + \sigma (x_1^3 - 3 x_1 y_1^2), \\
\dot{y}_3 &= (\mu_3 + d_{31}|z_1|^2 + d_{33}|z_3|^2)y_3 + \sigma (3 x_1^2 y_1 - y_1^3).
\end{align*}
\] (10)
A useful reduction in order results if Eqs. (8) are written in terms of the variables $z_j = a_j e^{i\phi_j}$, with real amplitudes $a_j > 0$ and phases $\phi_j$:

$$\begin{align*}
\dot{a}_1 &= (\mu_1 + d_{11}a_1^2 + d_{13}(x^2 + y^2))a_1 + a_1^3x, \\
\dot{x} &= (\mu_3 + d_{31}a_1^2 + d_{33}(x^2 + y^2))x + 3a_1y^2 + \sigma a_1^3, \\
\dot{y} &= (\mu_3 + d_{31}a_1^2 + d_{33}(x^2 + y^2))y - 3a_1xy.
\end{align*}$$

(12)

These equations avoid the complications associated with the singularity at $a_3 = 0$ present in Eqs. (11) because the standing solutions are contained within a single invariant plane $y = 0$. It is this fact which is responsible for removing the singular behavior of trajectories in (11) which pass through $a_3 = 0$. In the following we restrict attention, without loss of generality, to $\theta \in [0, \pi]$, i.e., to solutions with $y > 0$.

We observe that since all of the nonlinear terms are cubic, the existence of the various solutions can depend only on the coefficients $\sigma, d_{11}, \ldots, d_{33}$ and the ratio $\mu_1/\mu_3$. This is because Eqs. (8) are invariant under the scaling transformation

$$(\mu_1, \mu_3) \mapsto s^2(\mu_1, \mu_3), \quad z_j \mapsto sz_j, \quad t \mapsto t/s^2.$$ 

Thus all bifurcation sets in the $(\mu_1, \mu_3)$ plane are (segments of) straight lines through the origin. The validity of the truncation at cubic order requires that $|\mu_j| \ll 1$, but it is noteworthy that even complex behavior of Eqs. (8) such as heteroclinic cycles, period doubling and chaotic attractors can be found arbitrarily close to the origin $(\mu_1, \mu_3) = (0, 0)$.

3. Principal solutions

We now summarize the main types of simple equilibrium states. These solutions are described by Mizushima and Fujimura [9] and are also realized as special cases of the more general discussion given by Dangelmayr [3]. These fall into four categories.

1. **Trivial state** ($z_1, z_3 = 0$). The presence of this solution is forced by the symmetry $O(2)$. Its stability is specified by the two eigenvalues $\mu_1$ and $\mu_3$, each of which occurs with multiplicity 2.

2. **Pure modes** ($P_\theta$). The pure modes are given by $(z_1, z_3) = (0, \sqrt{-\mu_3/d_{33}} e^{i\theta})$ and exist when $\mu_3d_{33} < 0$. Their stability is determined by the eigenvalues $\lambda_3 = -2\mu_3$ and $\sigma_1 = \mu_1 - \mu_3d_{13}/d_{33}$. The first of these describes the response to perturbations in the amplitude $z_3$, while the second is of multiplicity 2 and describes the behavior associated with perturbations in the $z_1$ direction. There is also a zero eigenvalue due to the neutral stability of
the P modes with respect to translations, i.e., rotations \( \phi \rightarrow \phi + \text{const.} \) in the complex \( z_3 \) plane. As a result there is a circle of pure modes. Note that pure mode solutions of the type \((z_1, 0), z_1 \neq 0\), are not possible.

3. **Mixed modes** (MM\(_a\)). These take the form \((a_1 \exp(\imath \phi/3), \pm a_3 \exp(\imath \phi))\), where \( \phi \) again denotes an arbitrary spatial phase (of \( z_3 \)). In the representation (12) this arbitrary phase is eliminated and the MM states are characterized by \( a_1 \neq 0, x \neq 0 \) and \( y = 0 \). Specifically,

\[
x^2 = \frac{-\mu_1}{d_{11} r^2 + r + d_{13}}, \quad a_1 = r x, \tag{13}
\]

where \( r \) is a real root of the polynomial

\[
P(r) \equiv \sigma \mu_1 r^3 + (\mu_1 d_{31} - \mu_3 d_{11}) r^2 - \mu_3 r + (\mu_1 d_{33} - \mu_3 d_{13}). \tag{14}
\]

Thus there can exist at most three simultaneous circles of MM solutions.

4. **Traveling waves** (TW). Uniformly rotating waves are steady solutions of the system (11), although they are of course limit cycles of the original system (8), since both \( \dot{\phi}_1 \) and \( \dot{\phi}_3 \) (= \( 3\dot{\phi}_1 \)) are nonzero. The TW solutions are characterized by \( a_1 \neq 0, a_3 \neq 0 \), and \( \sin \theta 
eq 0 \). Thus

\[
\sigma a_1^2 + 3a_3^2 = 0, \tag{15}
\]

indicating that the TW states can only be present if \( \sigma = -1 \). In this case

\[
a_3^2 = \frac{-3(\mu_1 + \mu_3)}{9d_{11} + 3d_{13} + 3d_{31} + d_{33}}, \quad a_1 = \sqrt{3}a_3, \tag{16}
\]

\[
\cos \theta_{\text{TW}} = \frac{\mu_1(3d_{31} + d_{33}) - \mu_3(3d_{11} + d_{13})}{3\sqrt{3}\mu_1 + \sqrt{3}\mu_3}. \tag{17}
\]

The region in the \((\mu_1, \mu_3)\) plane in which the TW states exist is thus defined by \(-1 < \cos \theta_{\text{TW}} < 1 \) and \((3\mu_1 + \mu_3)(9d_{11} + 3d_{13} + 3d_{31} + d_{33}) < 0 \). We note that (excluding degenerate situations) such a region always exists when \( \sigma = -1 \). In contrast to the P and MM states the TW solutions are isolated.

### 4. Local bifurcations

The nature of the interactions between the different solutions as \( \mu_1 \) and \( \mu_3 \) are varied depends on the values of the coefficients \( \sigma, d_{11}, d_{13}, d_{31}, d_{33} \). In the following we discuss several of the more interesting choices in some detail. It is possible, nonetheless, to make some general statements about the possible bifurcation diagrams which result from traversing a given path in \((\mu_1, \mu_3)\) space. For this purpose it is convenient to identify all the P and MM states with a representative solution, chosen hereafter to be real.

The P branch bifurcates from the trivial state when \( \mu_3 = 0 \) in a primary pitchfork which is supercritical if \( d_{33} < 0 \) and subcritical if \( d_{33} > 0 \). The MM branch also bifurcates from the trivial state in a pitchfork bifurcation, this time at \( \mu_1 = 0 \). Although at onset this is a bifurcation to the \( z_1 \) mode, in the nonlinear regime an admixture of \( z_3 \) is always present. The bifurcation is supercritical if \( d_{11} < 0 \) and subcritical if \( d_{11} > 0 \). Near onset (\( \mu_1 = 0 \)) of the MM branch Eqs. (13) and (14) give

\[
x \approx \text{sign}(r) \left| \frac{\mu_1}{\mu_3 d_{11}} \right| \sqrt[2]{\frac{-\mu_1}{d_{11}}}, \quad r \approx \frac{\sigma \mu_3 d_{11}}{\mu_1}. \tag{18}
\]

and we see that the MM branch must change phase (reflected in the sign of \( x \)) between the half line \( \mu_1 = 0, \mu_3 > 0 \) and the half line \( \mu_1 = 0, \mu_3 < 0 \). This is accomplished by means of a transcritical bifurcation involving the P states.
and occurs at \( \mu_1 = (d_{33}/d_{33}) \mu_3 \). At this bifurcation the representative (real) MM state passes through \( x_1 = 0 \) with a corresponding \( \pi \) jump in the phase difference \( \theta \).

The MM branch typically displays saddle-node bifurcations as well. These occur when \( P(r) = P'(r) = 0 \). These two conditions lead to a quartic polynomial in \( \xi = \mu_1/\mu_3 \),

\[
L(\xi) \equiv a\xi^4 + b\xi^3 + c\xi^2 + d\xi + e,
\]
whose roots describe the half lines in the \((\mu_1, \mu_3)\) plane along which saddle-nodes occur. Here

\[
\begin{align*}
a &= 4d_{31}^3d_{33} + 27d_{33}^2, \\
b &= -4d_{13}^2d_{31}^3 - 54d_{13}d_{33} + 18\sigma d_{31}d_{33} - 12d_{11}d_{31}^2d_{33}, \\
c &= 27d_{13}^2 - 18\sigma d_{13}d_{31} - d_{31}^2 + 12d_{11}d_{13}d_{31}^2 - 18\sigma d_{11}d_{33} + 12d_{11}^2d_{13}d_{33}, \\
d &= -4\sigma + 18\sigma d_{11}d_{13} + 2d_{11}d_{31} - 54d_{13}^2d_{33} - 4d_{11}d_{13}, \\
e &= 4d_{11}^3d_{13} - d_{11}^2d_{33}.
\end{align*}
\]

Since a double root of any cubic polynomial must be real it follows that each real root of \( L(\xi) \) corresponds to a choice of \( \mu_1/\mu_3 \) for which there is a real double root of \( P(r) \). Requiring \( x^2 > 0 \) determines the sign of \( \mu_1 \) and leaves the possibility of having 0, 2, or 4 saddle-node bifurcations.

Hopf bifurcations may also occur on the MM branch, producing reflection-symmetric limit cycles representing standing waves (SW). Since this bifurcation is independent of the phase \( \phi \) of the MM, the bifurcation actually creates a circle of SW, hereafter \( \text{SW}_\phi \), i.e., a two-torus foliated by periodic orbits. A necessary condition for this Hopf bifurcation to occur is that the trace of the Jacobian of \((12)\) vanishes when restricted to the \( y = 0 \) subspace:

\[
r^3 + 2d_{11}r^2 + r + 2d_{33} = 0.
\]

Here \( r \) satisfies \( P(r) = 0 \) and is defined by \((13)\). We observe that if \( d_{11} < 0 \) and \( d_{33} < 0 \) (which will be the case throughout this paper) there are no solutions for \( r < 0 \) and consequently Hopf bifurcations may only occur when \( r > 0 \) (i.e., \( x > 0 \)). In the three-dimensional phase space we refer to this part of the MM branch as \( \text{MM}_0 \) (since \( \theta = 0 \)), while the other part is referred to as \( \text{MM}_\pi \) (since \( \theta = \pi \)). We also note that \( r \) passes through zero at the transcritical bifurcation and that in the system \((12)\) there is a discontinuity in \( x \) at this point. The frequency of the SW at onset is given by

\[
\omega^2 = 2d_{11}r^5 + 4r^4 + 2(3d_{13} - d_{31})r^3 + 4(d_{11}d_{33} - d_{13}d_{31})r^2 + 2d_{33}r.
\]

The TW solution exists only if \( \sigma = -1 \) and arises in a secondary pitchfork bifurcation from the MM branch when \( a_1 = \sqrt{3}a_2 \). This is a phase instability which breaks the reflection symmetry \( R \). Hopf bifurcations (torus bifurcations in the four-dimensional system) commonly occur on the TW branch and give rise to modulated traveling waves (MTW). At this bifurcation the frequency (of the modulation) can be expressed in terms of \( \cos \theta_{TW} \) as

\[
\omega_{TW}^2 = 18 \frac{9d_{11} + 3d_{13} + 3d_{31} + d_{33}}{d_{33} + 3d_{11} + 2\sqrt{3}\cos \theta_{TW}} (1 - \cos^2 \theta_{TW})
\]
\[
= 12(d_{11}d_{33} - d_{13}d_{31}) + 2\sqrt{3}(9d_{11} + d_{33} + 9d_{13} - 3d_{31}) \cos \theta_{TW} + 36(2 - \cos^2 \theta_{TW}).
\]

If there is a solution to the above equation the resulting value of \( \cos \theta_{TW} \) can be used in \((17)\) to determine the half line in the \((\mu_1, \mu_3)\) plane where MTW bifurcates from TW.

5. Geometric properties

We now briefly describe the geometry and physical interpretation of the phase space associated with the four-dimensional system \((8)\) and the three-dimensional system \((12)\).
In the four-dimensional system the most significant features are a result of the O(2) symmetry of these equations. The real subspace \((y_1 = y_3 = 0)\) is fixed by \(R\) and the action of \(T_\theta\) on this invariant two-dimensional subspace generates an \(S^1\) family of conjugate subspaces which are also fixed by reflection (about an appropriate origin). These subspaces contain both P and MM states, as well as any SW states, if present. In contrast the TW are periodic solutions in the four-dimensional system and exist (for \(\sigma = -1\)) in the subspace defined by \(a_1 = \sqrt{3}a_3\). For these solutions \(\phi_3 = 3\phi_1\). As a result the TW can be thought of as motion along the group orbit generated by the translations \(T_\theta\).

In the modified Cartesian system (12) as well as in the polar system (11) the translational symmetry has been factored out (recall that \(a_1, a_3\) and \(\theta\) are all invariant under \(T_\theta\)). The \(S^1\) family of \(R\) invariant planes in the four-dimensional system (8) thus collapses onto a single invariant plane \((y = 0)\) in the system (12), while each circle of MM states is mapped onto a single fixed point. The P states, however, are still represented by a (semi-)circle of fixed points \(P_\theta\) in the three-dimensional system (12) since they are defined by \(x^2 + y^2 = -\mu_3/d_{33}\). This (semi-)circle should not be confused with the circle \(P_\theta\) of pure modes in the four-dimensional system because the remaining \(S^1\) degree of freedom is transverse to the group orbit of \(T_\theta\) (in the four-dimensional system) and is a result of the double multiplicity of the P eigenvalue in the \(z_1\) direction (this degeneracy is absent in the 1:2 mode interaction). One could imagine (again in the four-dimensional system) sitting at a particular P state with a fixed value of \(\phi_3\) in the \(z_3\) plane and performing a rotation in the \(z_1\) plane. Because the eigenvalues in the \(z_1\) plane are degenerate this rotation is associated with a zero eigenvalue. The surviving \(S^1\) degree of freedom for the P states in (11) and (12) can be thought of as a rotation of this type. Thus the (semi-)circle \(P_\theta\) does not represent a family of distinct points in the sense that \(P_\theta\) does but should be thought of in terms of the different possible orientations (in the \(z_1\) plane) of a representative member of \(P_\theta\). Nonetheless, for simplicity, we will hereafter refer to the members of \(P_\theta\) as if they were distinct solutions.

6. Numerical results

In this section we present detailed results for five choices of the coefficients \(d_{ij}\), assuming always that \(\sigma = -1\), i.e., that TW are possible. The most direct physical significance belongs to the set of coefficients referred to below as case B. These values correspond to two-dimensional Rayleigh–Bénard convection with no-slip fixed temperature boundary conditions at the top and bottom in the limit of small Prandtl number, and were obtained by Mizushima and Fujimura [9]. The remaining sets (A, C–E) were chosen so as to produce variations on the dynamics present in case B. In all the cases considered,

\[
\begin{align*}
  d_{11} &< 0, \\
  d_{33} &< 0, \\
  d_{13} + d_{31} &< 2\sqrt{d_{11}d_{33}},
\end{align*}
\]

so that Eqs. (12) have an absorbing ball around the origin. A typical situation is shown in Fig. 1. This figure shows the codimension-1 bifurcation lines identified in Section 4 for a set of coefficient values referred to hereafter as case A. The diagram agrees with the general analysis given by Dangelmayr [3] and Armbruster et al. [10], although additional codimension-1 bifurcations are included. Throughout this paper we follow Armbruster et al. and draw representative bifurcation diagrams along clockwise circular paths (of unit radius) centered at the origin \((\mu_1, \mu_3) = (0, 0)\); see Fig. 2a. The resulting range within which the P, MM, TW, MTW, and SW states are present is indicated in Fig. 1 by arcs; heavy arcs indicate that the solutions are stable. The origin \((z_1, z_3) = (0, 0)\) is stable in the third quadrant. Observe that there is a small interval of bistability between stable P and MM states in the second quadrant due to a subcritical bifurcation from P to MM\(_0\). The MM\(_0\) state is stable between a saddle-node line in this quadrant and a line of Hopf bifurcations to SW. The MM\(_0\) branch undergoes a second saddle-node bifurcation in the first quadrant before terminating on the trivial state in a primary bifurcation along the half line \((0, \mu_3), \mu_3 > 0\).
Stable TW are born in a parity-breaking steady-state bifurcation from the stable MM$_0$ branch in the first quadrant and lose stability in a torus bifurcation to MTW in the second quadrant. Additional features of this diagram are discussed below.

6.1. Case A: $d_{11} = -0.633$, $d_{13} = 1.663$, $d_{31} = -1.649$, $d_{33} = -0.355$

Fig. 2a shows the bifurcation diagram for case A obtained by traversing a closed clockwise circular path in the unfolding plane (Fig. 1) beginning in the third quadrant. The diagram shows $a_3$ versus the polar angle $\alpha$ in the $(\mu_1, \mu_3)$ plane (see Fig. 1). The Hopf bifurcation to SW on the MM$_0$ branch (at $\alpha = 2.1824$) is subcritical, while the torus bifurcation to MTW on the TW branch is supercritical. The bifurcation diagram indicates that the resulting MTW branch loses stability via a (subcritical) torus bifurcation before merging with SW. As the latter bifurcation is approached we find numerically that one of the torus frequencies vanishes as the square root of the distance from the bifurcation but that no complex dynamics are directly due to this bifurcation. The SW itself is ultimately destroyed in another global bifurcation when it forms a heteroclinic connection between $P_0$, a pure mode (with $\theta = 0$) in the $y = 0$ invariant plane, and the origin $O$. This global bifurcation occurs in all the cases considered and a straightforward argument shows that the resulting heteroclinic cycle is always unstable to perturbations out of the $y = 0$ plane (since $\mu_3 > 0$), while its stability within the $y = 0$ plane depends on the ratio $\nu \equiv \mu_1 \sigma_3 / \mu_3 \sigma_1$ (see Section 7). In all the cases we consider $\nu < 1$ when the heteroclinic connection $P_0 \rightarrow O$ forms, and consequently the SW must become completely unstable before it can disappear in such a bifurcation.

In the region near the Hopf bifurcation to SW we have found evidence for complex dynamics of Shil’nikov type (see Fig. 3). These chaotic solutions are associated with a cascade of periodic solutions of Eqs. (12) whose period $\tau$ is shown in Fig. 2b as a function of $\alpha$. Each of these solutions corresponds to a quasi-periodic solution of the original system (8). Within the three-dimensional system (12) these solutions can be followed numerically and we have done so using a version of AUTO included with the dynamical systems software XPPAUT [16].
Fig. 2. Case A: (a) bifurcation diagram \(a_3(\alpha)\), where \(\alpha\) is the polar angle from Fig. 1; (b) (quasi-)periodic solutions shown in terms of the period \(\tau\) in the 3D system (12); (c) an enlargement of isola I16 showing clearly the 'twist region', characterized by a substantial drop in the period \(\tau\).

For the parameters of case A these periodic orbits form a cascade of isolas referred to, from the bottom up, as I1, I2, \ldots. These appear to accumulate on the interval \(2.194 \lesssim \alpha \lesssim 2.252\), throughout which SW are present. At each \(\alpha\) in this range there is an (apparently) infinite sequence of (pairs of) periodic orbits with period tending to infinity, indicating the presence of a (structurally stable) global connection. Each isola appears in a saddle-node

Fig. 3. Stable chaotic orbit associated with isola I5 of case A at \(\alpha = 2.0796\).
bifurcation at the left and disappears in another saddle-node bifurcation on the right. The isolas are spaced apart by roughly the period of the SW suggesting that the orbits in each family (isola) differ asymptotically by one SW orbit. The period of the SW does not give the precise separation of the isolas, however, since the periodic orbits also visit the neighborhood of several fixed points and there is a corresponding change in the period due to the varying amount of time spent there.

Observe that with increasing period the isolas rapidly develop a characteristic twist near the right saddle-node; the twist introduces at least two additional saddle-node bifurcations onto the isola. As a result if one traverses the isola from the leftmost saddle-node in a clockwise direction one finds that initially both nontrivial Floquet multipliers are real, positive and outside the unit circle; these collide on the real axis, move into the complex plane and collide again on the negative real axis, still outside the unit circle. With decreasing $\alpha$ one enters the unit circle at $-1$, triggering a cascade of period-doubling bifurcations. These are followed by a reverse cascade that terminates when the second Floquet multiplier enters the unit circle, stabilizing the period-1 orbits just before the second saddle-node bifurcation at the right. Before this bifurcation is possible the negative Floquet multipliers must again become complex, and collide on the positive axis, creating two positive Floquet multipliers inside the unit circle, one of which then exits the unit circle at $+1$ at the second saddle-node and re-enters it again at the third saddle-node (see Fig. 4). What happens on the twisted part of an isola varies. For the lower isolas the branch between the third and fourth saddle-nodes often contains no additional bifurcations and is stable. For higher isolas the situation becomes progressively more complex. There are often period-doubling bifurcations just below the third and fourth saddle-nodes which destabilize the period-1 branch (see Figs. 4 and 5). In such cases there are also reverse period doublings which restabilize the period-1 branch before the point of minimum period. For isolas with even higher period it becomes increasingly difficult to track reliably all the bifurcations on the twisted part of the isola as it narrows and becomes almost vertical. Additional saddle-node, period-doubling, and even torus bifurcations are almost certainly present. Finally, the period-1 orbits between the first and last saddle-node bifurcations are unstable and no additional bifurcations occur.

As a result, if one follows a stable period-1 solution leftwards beginning at the period minimum, one finds that it loses stability either at the saddle-node or in a period-doubling cascade. Fig. 3 shows a stable chaotic orbit found after such a procedure. This attractor may be connected to the first (‘upper’) period-doubling cascade. Alternatively, the system may jump onto the stable portion of a higher period isola, and so on. These transitions are hysteretic.
The movement of the Floquet multipliers along isola I5 is shown in Fig. 4. We consider this isola to be both relatively simple and representative of moderate period isolas. The small insets show the location in the complex plane of the two nontrivial Floquet multipliers at various locations around this isola. The periodic orbit is stable when both multipliers, indicated by solid dots, lie inside the unit circle. The approximate locations of the first period-doubling bifurcations are indicated by small semi-circles on the isola whose orientation shows the direction of the subsequent period-doubling cascade. Note that the only stable solutions are found either near the rightmost saddle-nodes, along the bottom of the twisted part of the isola, or just below the left saddle-node on the twisted part of the isola. Away from the saddle-nodes each solution loses stability via period-doubling and the resulting cascades, if completed, can result in stable chaos. Fig. 5 shows half the period of the resulting period-doubled branches, along with the period of the period-1 solutions. Note that a period-doubled branch connects the period-doubling bifurcation near the left and right saddle-nodes on the main part of the isola and replicates the main behavior of the period-1 isola, including the twist, making several excursions along both sides of it. There is likewise a connection between the two period-doubling bifurcations on the right side of the twist (and also for those on the left side), but while these asymptote to the same structure they exhibit no large oscillations of the type shown by the first period-doubled branch. Together these results suggest that as $\tau \to \infty$ the structure of the isola, although complicated, has a strong element of universality. Fig. 6 shows the structure of the period-1 orbits along I9. These longer period orbits approximate well the global connections formed in the limit $\tau \to \infty$, and reveal a complex sequence of transitions that forms the subject of Section 7. Note, however, that any exact connection involving the SW can only be present for $\alpha > 2.1824$ (cf. Fig. 2b).

6.2. Case B: $d_{11} = -0.637$, $d_{13} = 0.971$, $d_{31} = -1.494$, $d_{33} = -0.443$

Case B corresponds to the coefficient values computed by Mizushima and Fujimura [9] for Rayleigh–Bénard convection with no-slip boundary conditions at the top and bottom in the small Prandtl number limit. The corresponding results are shown in Figs. 7a and b and are similar to case A. The higher period isolas again show a dramatic decrease in period in the twist region, and appear to accumulate on the interval $1.945 \lesssim \alpha \lesssim 1.983$ as $\tau \to \infty$. 
6.3. Case C: $d_{11} = -0.323$, $d_{13} = 1.85$, $d_{31} = -1.35$, $d_{33} = -0.721$

The results for case C are shown in Figs. 8a and b. The SW bifurcation is now supercritical but nearly degenerate. The SW first lose stability to MTW in a symmetry-breaking bifurcation and the second nontrivial Floquet multiplier subsequently exits the unit circle at a saddle-node bifurcation before the SW branch again terminates in a heteroclinic connection between $P_0$ and $O$. In this case the SW exist only over a very narrow range of parameters which does not appreciably overlap the region in which we find the family of isolas. This time there is no torus bifurcation on the
MTW branch which remains stable between the initial Hopf bifurcation and its merger with SW. Also by contrast to cases A and B it is no longer clear whether or not the cascade of isolas accumulates on a continuous interval of finite length. In fact the higher period isolas in this case deform substantially and there is reason to believe that they either cease at finite period or continue to deform (possibly breaking up) as \( \tau \to \infty \) so that the limiting parameter set is either discrete or an extremely narrow interval (see Section 7).

6.4. Case D: \( d_{11} = -0.245, d_{13} = 1.633, d_{31} = -4.46, d_{33} = -0.43 \)

The results for this choice of coefficients are shown in Figs. 9a and b, and differ substantially from the ones already described. The Hopf bifurcation to SW is supercritical but the SW branch turns around almost immediately, forming unstable standing waves. Thus away from the Hopf bifurcation the situation is much as in cases A and B, with the MTW losing stability at a torus bifurcation before terminating on the SW branch. However, although a hierarchy of isolas is still present, Fig. 9b shows that the situation is now much more complicated. While the lower four periodic families do indeed form isolas of the type already seen, the next opens out, with a single periodic orbit whose period tends to infinity on the right, and a complicated zigzag structure in the \( \tau(\alpha) \) curve arising from repeated mergers between adjacent isolas on the left. This entire process is similar to the type of merging referred to in [17] as a ‘zipper’, and observed earlier by Zimmermann et al. [18]. With increasing period these mergers cease, and a Shil’nikov-like oscillation closes the branch at the left, leaving a hierarchy of isolas with increasing period, as in cases A and B. The overall effect of the mergers is to produce an orbit whose period is infinite at \( \alpha \approx 1.95206 \) and decreases along the oscillations in the \( \tau(\alpha) \) curve, before increas-
ing and becoming infinite again at $\alpha \simeq 1.70829$. The latter infinite period orbit corresponds to the formation of a homoclinic connection involving the TW state produced in a parity-breaking bifurcation from MM$_0$ (see Fig. 10).

The possibility of such a connection is familiar from the analysis of the Takens–Bogdanov bifurcation with O(2) symmetry [19], another system defined on $\mathbb{C}^2$ and with the same symmetry. Note that no similar connections involving the MM are possible. For TW such a connection is prohibited by the invariance of the $y = 0$ plane which contains the MM$_0$ state and its two-dimensional unstable manifold, while for $\alpha < \alpha_{TW}$ MM$_0$ is repelling in all directions within the three-dimensional system (12). Here $\alpha_{TW}$ is the value of $\alpha$ at which the parity-breaking bifurcation from MM$_0$ takes place. For the coefficients used $\alpha_{TW} \simeq 1.70830$, while the period $\tau$ reaches $\tau \approx 10^6$ at $\alpha \simeq 1.70829$, i.e., when the TW are present.

In contrast the infinite period orbit at $\alpha \simeq 1.95206$ corresponds to a heteroclinic cycle involving the MM$_0$ state, the origin O, and a pure mode $P_\beta$ not in the invariant $y = 0$ plane (see Fig. 11). The remaining isolas (as in case C) do not appear to accumulate on a (large) finite interval of $\alpha$ as $\tau \to \infty$.

Both the isolas and the large amplitude zigzags arising from the mergers of adjacent isolas exhibit numerous additional saddle-node bifurcations. As a result the number of intervals in which stable periodic solutions can be found is much larger, though such intervals remain small.

6.5. Case E: $d_{11} = -0.162, d_{13} = 1.629, d_{31} = -2.159, d_{33} = -0.2$

This case is similar to case C in that the Hopf bifurcation to SW is supercritical, the SW transfer stability to the MTW in a parity-breaking bifurcation and then turn around at a saddle-node bifurcation before terminating in a
heteroclinic connection between $P_0$ and $O$. The appearance of the isolas (Fig. 12b) differs substantially from the preceding cases; there are no mergers between adjacent isolas, and as in cases C and D the isolas do not appear to limit on a finite interval of $\alpha$ as $\tau \to \infty$.

Fig. 9. Case D: (a) bifurcation diagram, (b) (quasi-)periodic solutions, (c) enlargement of a high period isola with its complicated 'twist' region.

Fig. 10. Homoclinic connection to TW at $\alpha \approx 1.70829$. 

Fig. 11. (a) Heteroclinic cycle $O \rightarrow P \rightarrow MM_0 \rightarrow O$ at $\alpha \simeq 1.95206$ and (b) associated Shil'nikov type oscillations in $\tau(\alpha)$.

Fig. 12. Case E: (a) bifurcation diagram, (b) (quasi-)periodic solutions, (c) enlargement of isola I7.
7. Interpretation and analysis

In this section we discuss in more detail the origin of the behavior described in Section 6. Our interpretation assumes that this behavior is in some sense typical. Support for this belief is provided in Fig. 13 which shows that the isola I5 for case B bears substantial similarity to the corresponding isola for case A (Fig. 5). In all the cases described the complex behavior occurs near the Hopf bifurcation to SW on the MM0 branch. In this parameter regime the origin is unstable to the pure modes $P_0$ but is still stable in the $a_1$ direction. In the phase space of (12) $O$ thus has a one-dimensional stable manifold $W^s(O)$ contained within the invariant $y = 0$ plane (see Fig. 14). It is also easy to see that since $\dot{x} \leq 0$ when $x = y = 0$, $W^s(O)$ must lie on the positive $x$ side of this plane where the possible (finite amplitude) limit sets (as $t \to -\infty$) are MM0, SW (when present), and $P_0$. A generic situation then is that $W^s(O)$ either lies within the two-dimensional unstable manifold of MM0 or that of SW (when present). We hereafter denote these manifolds (which are contained within the $y = 0$ plane) as $W^u(MM_0)$ and $W^u(SW)$, respectively.

![Fig. 14. Sketch of phase space when SW are present.](image)
unstable manifold of O, $W^u(O)$, is two-dimensional and contained within the invariant plane $a_1 = 0$; $W^a(O)$ thus connects (persistently) to the (semi-)circle of pure modes, $P_\theta$.

The MM$_\theta$ state lies in the region $x < 0$, $y = 0$ and in this regime has a one-dimensional unstable manifold, $W^u(MM_\theta)$, transverse to the $y = 0$ plane and a two-dimensional stable manifold $W^s(MM_\theta)$ contained within this plane and associated with a complex conjugate pair of eigenvalues. There are two P states which lie in the $y = 0$ plane, $P_0$ and $P_\pi$, whose unstable manifolds, $W^u(P_0)$ and $W^u(P_\pi)$, are also contained within this plane and hence must fall, generically, within $W^s(MM_\theta)$. It is also true that $W^u(MM_0)$ (or $W^u(SW)$ when SW are present) is captured (except for a set of measure zero) by $W^s(MM_\theta)$. Finally, since in the region of interest the pure modes have lost stability in the $a_1$ direction due to the transcritical bifurcation to MM$_0$ we may consider the union of the unstable manifolds of the $P_\theta$ family as a two-dimensional invariant manifold which we refer to as $W^u(P_\theta)$.

We thus find that there are several types of structurally stable heteroclinic connections in the parameter regime of interest: $MM_0$ (or SW) $\rightarrow$ O, O $\rightarrow$ $P_\theta$, $P_0$ $\rightarrow$ MM$_\pi$, $P_\pi$ $\rightarrow$ MM$_\pi$, and MM$_0$ (or SW) $\rightarrow$ MM$_\pi$. The persistence of these connections over a finite range of parameters is a consequence of the symmetries (1) which are responsible for the invariance of the $y = 0$ and $a_1 = 0$ planes, while the fact that there is an entire (semi-)circle of P states which connect to O is a consequence of the degeneracy of the eigenvalues of the P states in the four-dimensional system (8).

There are two distinct scenarios depending on whether or not the SW are present and we discuss first the case when they are (cf. Fig. 14). In this case, since $W^u(SW)$ is two-dimensional, one can get a codimension-1 global bifurcation at which $W^u(SW)$ contains $W^u(MM_\theta)$. In the following we assume that this occurs when $\alpha$ takes a particular value, $\alpha_\pi$, say. At $\alpha_\pi$ several different kinds of heteroclinic cycles are created simultaneously: $O \rightarrow P_\pi \rightarrow MM_\pi \rightarrow SW \rightarrow O$, $O \rightarrow P_0 \rightarrow MM_\pi \rightarrow SW \rightarrow O$, and $SW \rightarrow MM_\pi \rightarrow SW$ (and combinations). Note that there is an entire continuum of heteroclinic cycles of the last kind. A connection of this type is quite clearly present in the limit $\tau \rightarrow \infty$ for locations at the bottom of the twist region of each isola (see Fig. 6e). As discussed further below the global bifurcation at $\alpha_\pi$ is in fact responsible for the complexity of the twist region found in Section 6 which is a prominent feature of the isolas for all of the parameter sets investigated.

There is another type of global connection that is essential for understanding the structure of the isolas. This connection is present if the two-dimensional manifold $W^u(P_\theta)$ intersects transversely the tubular manifold $W^s(SW)$, and is the result of a global bifurcation that occurs when these two manifolds first touch as $\alpha$ decreases. This initial tangency corresponds to the accumulation point of the leftmost saddle-nodes on the isolas. With further decrease in $\alpha$ the manifolds intersect transversely, and for typical $\alpha$ values there are at least two P states, $P_{\theta_j}$ ($j \geq 2$ and even), connected to SW by a heteroclinic orbit. The angles $\theta_j$ are of course functions of the parameter $\alpha$ and consequently these heteroclinic connections ‘slide’ along the family of P states as $\alpha$ varies. Figs. 6b–d and f–h show only one part of these connections since each figure describes a periodic orbit either on the top or on the bottom of the isola. In the limit $\tau \rightarrow \infty$ one must imagine gluing together the orbits shown, for example, in Figs. 6b and f–h to form an infinite period orbit of this type. Since the connections $SW \rightarrow O$ and $O \rightarrow P_\theta$ already exist the intersection of $W^u(P_\theta)$ and $W^s(SW)$ generates persistent heteroclinic cycles, namely $O \rightarrow P_{\theta_j} \rightarrow SW \rightarrow O$. This scenario provides a qualitative explanation of the appearance of the isolas away from the twist region when SW are present. We remark that (at least in cases A and B) the connections at the rightmost saddle-nodes involve pure modes $P_{\theta_j}$ that are close to either $P_0$ (lower saddle-node) or $P_\pi$ (upper saddle-node); see Fig. 6.

The second case we need to consider arises when the SW are absent. In this situation one does not expect to see the intersection of $W^u(MM_\theta)$ with $W^s(MM_\theta)$ since both are one-dimensional manifolds and such an intersection would be a codimension-2 phenomenon. The other global bifurcation that can occur (and is of codimension-1) arises when $W^u(P_\theta)$ contains $W^s(MM_\theta)$. However, the resulting heteroclinic cycle, $O \rightarrow P_{\theta_j} \rightarrow MM_0 \rightarrow O$, involves only one particular P state (see Fig. 11), denoted by $P_{\theta_j}$, and hence is structurally unstable. In fact, one expects at most a finite number of parameter values at which such cycles are present, instead of the continuous family of
heteroclinic cycles involving $P_0$ which are present in the first case. In a situation without SW a cascade of isolas could not limit on a finite interval of $\alpha$ but would either cease at a finite period or shrink to zero width, possibly breaking apart into a finite number of Shil'nikov bifurcations, cf. [20].

In what follows we construct Poincaré maps for the case where heteroclinic cycles exist in the presence of SW. Although this is not the only case of interest, applying rigorously only to cases A and B, we find that the low period orbits are sufficiently far from the asymptotic limit to be insensitive to the presence of SW. It is not too surprising then that the lower period isolas have a similar structure even for parameter values for which the global connection that is present does not in fact involve SW. In this sense the maps derived below are useful in explaining the qualitative features of the isolas even in cases C–E. There are three distinct regions on a typical isola: the part without the twist, the part of the twist where the period drops dramatically, and the stable portion of the twist where the period is a minimum. These must each be considered separately since (in the presence of SW) they asymptote to distinct kinds of heteroclinic cycles. These regions are therefore described by distinct Poincaré maps, hereafter referred to as $T_1$, $T_2$ and $T_3$.

7.1. Poincaré map 1: $O \rightarrow P_\theta \rightarrow SW \rightarrow O$

We begin with the map $T_1$. This map describes the situation where $W^u(P_\theta)$ intersects $W^s(SW)$ generating (structurally stable) heteroclinic cycles of the type $O \rightarrow P_\theta \rightarrow SW \rightarrow O$. The subsequent analysis focuses on one such representative cycle at a fixed value of the parameter $\alpha$ and assumes $\sin \theta \sim 1$.

We begin by linearizing Eqs. (11) near the origin to get

$$\dot{a}_1 = \mu_1 a_1, \quad \dot{a}_3 = \mu_3 a_3, \quad \dot{\theta} = 0,$$

with $\mu_1 < 0$, $\mu_3 > 0$, and choose the Poincaré sections (see Fig. 15)

$\Sigma^1_O : \{(a_1, x, y) | a_1 = \epsilon, \ r \leq \epsilon\}, \quad \Sigma^2_O : \{(a_1, r, \theta) | 0 \leq a_1 \leq \epsilon, \ r = \epsilon\},$

Fig. 15. Sketch of the Poincaré sections defined in Sections 7.1–7.3.
where \( r = a_3 = \sqrt{x^2 + y^2} \). The time of flight from \( \Sigma_O^1 \) to \( \Sigma_O^2 \) is given by
\[
\tau_O = \frac{1}{\mu_3} \log \left( \frac{x}{r} \right),
\]
and it is straightforward to obtain the map \( T_O : \Sigma_O^1 \rightarrow \Sigma_O^2 \):
\[
T_O : \begin{pmatrix} \epsilon \\ r \cos \theta \\ r \sin \theta \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ \frac{\mu_1}{\mu_3} \epsilon \\ \frac{\epsilon}{\theta} \end{pmatrix}.
\]
(24)

Next we linearize Eqs. (11) about the circle of P states, obtaining
\[
\dot{a}_1 = \sigma_1 a_1, \quad \dot{\xi} = \sigma_3 \xi, \quad \dot{\theta} = -3 \left( \frac{-\mu_3}{d_{33}} \right)^{1/2} \sin \theta a_1,
\]
(25)
where \( \sigma_1 > 0, \sigma_3 < 0. \)

Here \( \xi \equiv (-\mu_3/d_{33})^{1/2} - r \), with the quantities \( \sigma_j \) defined in Section 3. In the following the term \( \sin \theta \) in the \( \dot{\theta} \) equation may be treated as constant (to leading order), resulting in convenient simplifications. After choosing the Poincaré sections,
\[
\Sigma_P^1 : \{(a_1, \xi, \theta)| 0 \leq a_1 \leq \epsilon, \xi = \epsilon\}, \quad \Sigma_P^2 : \{(a_1, \xi, \theta)| a_1 = \epsilon, 0 \leq \xi \leq \epsilon\},
\]
we find the time of flight
\[
\tau_P = \frac{1}{\sigma_1} \log \left( \frac{\epsilon}{a_1} \right),
\]
(27)
and the map \( T_P : \Sigma_P^1 \rightarrow \Sigma_P^2 \):
\[
T_P : \begin{pmatrix} a_1 \\ \epsilon \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ \frac{\epsilon}{\sigma_1 a_1} \\ \theta - \frac{3}{\sigma_1} \left( \frac{-\mu_3}{d_{33}} \right) (\epsilon - a_1) \sin \theta \end{pmatrix}.
\]
(28)
Near the SW one can use appropriate normal coordinates in which the SW limit cycle is circular and which satisfy (to leading order)
\[
\dot{\hat{y}} \left( t + \frac{2\pi}{\omega} \right) = e^{2\pi \lambda/\omega} \dot{\hat{y}}(t), \quad \dot{z} \left( t + \frac{2\pi}{\omega} \right) = e^{2\pi \rho/\omega} z(t), \quad \dot{\vartheta} \left( t + \frac{2\pi}{\omega} \right) = \vartheta(t) - 2\pi = \vartheta(t).
\]
Here \( \lambda \) and \( \rho \) are the Floquet exponents of the SW limit cycle and \( \omega \) is its frequency. We remark that \( \lambda \) and \( \rho \) must be real due to the symmetry \( R \) and that in the case of interest \( \lambda < 0 \) and \( \rho > 0 \). Upon defining Poincaré sections,
\[
\Sigma_{SW}^1 : \{(\hat{y}, z, \vartheta)| \hat{y} = \epsilon, 0 \leq z \leq \epsilon\}, \quad \Sigma_{SW}^2 : \{(\hat{y}, z, \vartheta)| 0 \leq \hat{y} \leq \epsilon, z = \epsilon\},
\]
one obtains a map \( T_{SW} : \Sigma_{SW}^1 \rightarrow \Sigma_{SW}^2 \) valid when the number of SW oscillations is large (i.e., \( z \) is small):
\[
T_{SW} : \begin{pmatrix} \epsilon \\ z \\ \vartheta \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \left( \frac{\lambda}{\rho} \right) \\ \frac{\lambda}{\rho} \log \frac{\epsilon}{z} \\ \vartheta - \frac{\omega}{\rho} \vartheta \end{pmatrix}.
\]
(29)
The global maps are obtained by linearizing about the heteroclinic cycle. For this purpose we denote \( W^u(P_{\theta_0}) \cap W^s(SW) \cap \Sigma_{SW}^1 \) by \((\tilde{y}, z, \theta) = (\epsilon, 0, \theta_0)\) and \( W^s(0) \cap \Sigma_{SW}^2 \) by \((\hat{y}, \hat{z}, \hat{\theta}) = (0, \epsilon, \hat{\theta}_0)\). Making use of the invariance of the \( y = 0 \) and \( a_1 = 0 \) planes we then obtain the maps \( T_{O \rightarrow P} : \Sigma_O^2 \rightarrow \Sigma_P^1 \), \( T_{P \rightarrow SW} : \Sigma_P^1 \rightarrow \Sigma_{SW}^2 \), and \( T_{SW \rightarrow O} : \Sigma_{SW}^2 \rightarrow \Sigma_O^1 \):

\[
T_{O \rightarrow P} : \begin{pmatrix} a_1 \\ \epsilon \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} b^2_1 \epsilon a_1 \\ \epsilon \\ \theta + c_1 a_1 \end{pmatrix},
\]

(30)

\[
T_{P \rightarrow SW} : \begin{pmatrix} \epsilon \\ \xi \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ d_j (\theta - \hat{\theta}_j) + e_j \xi \\ \theta_j + f_j d_j (\theta - \hat{\theta}_j) + g_j \xi \end{pmatrix},
\]

(31)

\[
T_{SW \rightarrow O} : \begin{pmatrix} \epsilon \\ \hat{\theta} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ h \hat{y} + k (\theta - \hat{\theta}_0) \\ l^2 \hat{y} \end{pmatrix}.
\]

(32)

Here \( \hat{\theta}_j \) denotes \( W^u(P_{\theta_0}) \cap \Sigma_P^1 \) and is given by

\[
\hat{\theta}_j = \theta_j - \tilde{a} \sin(\theta_j), \quad \tilde{a} \equiv \frac{3 \epsilon}{\sigma_1} \left( \frac{-\mu_3}{d_{33}} \right)^{1/2}.
\]

(33)

By taking the composition \( T_{SW \rightarrow O} \circ T_{SW} \circ T_{P \rightarrow SW} \circ T_P \circ T_{O \rightarrow P} \circ T_O \) one obtains a map \( T_1 : \Sigma_O^1 \rightarrow \Sigma_O^1 \). The fixed points of this map are solutions of

\[
r \cos \theta = \tilde{h} z(r, \theta)^{-\lambda_j/\rho} + k \Phi(r, \theta),
\]

(34)

\[
r \sin \theta = l^2 z(r, \theta)^{-\lambda_j/\rho},
\]

(35)

where \( z(r, \theta) \) is the coordinate on \( \Sigma_{SW}^1 \) perpendicular to \( W^s(SW) \),

\[
z(r, \theta) = d_j (\theta - \tilde{\theta}_j + \tilde{c}_j r^{-\mu_1/\mu_3} - \tilde{a}(1 - \tilde{b}_j r^{-\mu_1/\mu_3}) \sin(\theta + \tilde{c}_j r^{-\mu_1/\mu_3})) + \tilde{c}_j r^{\nu},
\]

(36)

and \( \Phi(r, \theta) \) is the angle on \( \Sigma_{SW}^2 \) relative to \( W^s(O) \cap \Sigma_{SW}^2 \). Here \( \nu \equiv (\mu_1 \sigma_3 / \mu_3 \sigma_1) > 0 \) as in Section 6, and

\[
\tilde{b}_j^2 = b_j^2 e^{\mu_1/\mu_3}, \quad \tilde{c}_j = c_j e^{1+\mu_1/\mu_3}, \quad \tilde{e}_j = e_j e^{1-\nu b_j^{-2\sigma_3/\sigma_1}}, \quad \tilde{h} = h e^{1+\lambda_j/\rho}, \quad \tilde{L} = l^2 e^{1+\lambda_j/\rho}.
\]

(37)

Such solutions exist provided \( z > 0 \) (so that the trajectory has a chance of returning to \( \Sigma_O^1 \)) and \( \Phi(r, \theta) \) is small. Since a given trajectory may perform a number of SW oscillations before finally crossing \( \Sigma_{SW}^2 \) near \( \theta = \theta_0 \) this requires that \( \Phi \) in (34) is defined mod \( 2\pi \). Consequently we may write

\[
\Phi(r, \theta) - 2\pi n = \theta_j - \theta_0 + \tilde{g}_j r^{\nu} + f_j z(r, \theta) + \frac{\omega}{\rho} \log \left( \frac{z(r, \theta)}{\epsilon} \right),
\]

(38)

where

\[
\tilde{g}_j = e^{1-\nu b_j^{-2\sigma_3/\sigma_1}} (g_j - e_j f_j),
\]

(39)
and we consider the right-hand side of (38) to be a continuous function taking values on the real line. As a result we can associate with each isola an integer \( n \). Since \( \theta \approx \theta_j \) we now solve Eq. (35) for \( z \), obtaining from (34) and (38) the single equation

\[
2\pi n + \psi_j + \tilde{k}_j r + \tilde{g}_j r^\nu + \tilde{f}_j r^{-\rho/\lambda} - \frac{\omega}{\lambda} \log r = 0, \tag{40}
\]

where we have written

\[
\psi_j = \theta_j - \theta_0 - \frac{\omega}{\rho} \log \epsilon - \frac{\omega}{\lambda} \log(\tilde{I}^{-2} \sin \theta_j), \quad \tilde{k}_j = k^{-1}(\tilde{hI}^{-2} \sin \theta_j - \cos \theta_j), \quad \tilde{f}_j = f_j(\tilde{I}^{-2} \sin \theta_j)^{-\rho/\lambda}.
\]

The solutions of Eq. (40) correspond to period-1 solutions of the three-dimensional system (12) and when considered as functions of the bifurcation parameter \( \alpha \) describe the structure of the bifurcation diagrams in Figs. 2b and 3b, at least within the region where SW are present and \( \sin \theta_j \approx 1 \). Note that it may be necessary to allow \( \theta_j - \theta_0 \) to take values outside of \( [-\pi, \pi] \) in order to label each isola by a constant value of \( n \).

Solutions of Eq. (40) can be used to determine \( \theta \) from Eq. (35). Since there is a separate equation for each \( n \) we expect a countable infinity of solutions, at least for values of \( \alpha \) within a certain range. In this range when \( r \) is small (\( n \) large), Eq. (40) yields

\[
r_n = r_j e^{2\pi (\lambda/\omega)n}, \quad r_j \equiv e^{(\lambda/\omega)\psi_j}. \tag{41}
\]

This expression in conjunction with (24) and (27) in turn implies that at fixed \( \alpha \) the difference in period between corresponding solutions on successive isolas is given by

\[
\tau_{n+1} - \tau_n = \frac{2\pi}{\omega} \left( 1 + \left| \frac{\lambda}{\mu_3} \right| + \frac{\lambda \mu_1}{\sigma_1 \mu_3} \right). \tag{42}
\]

The first term comes from the extra SW oscillation, while the next two provide the correction due to the increased time spent near O and P\(_j\), respectively. The predictions of this (asymptotic) expression (see Fig. 16) are in excellent agreement with the observed separation between successive isolas estimated from Figs. 2b and 7b to be approximately 7.95 (case A) and 8.27 (case B).

Fig. 16. Plot of \( \tau_{n+1} - \tau_n \) predicted by Eq. (42) over the range of \( \alpha \) values where SW and isolas coexist for (a) case A and (b) case B.
7.2. Poincaré map 2: \( O \rightarrow (P_0, P_\pi) \rightarrow MM_\pi \rightarrow SW \rightarrow O \)

Fig. 6 shows that as a function of \( \alpha \) the heteroclinic cycles deform continuously into ones involving the special points \( P_0 \) or \( P_\pi \). This situation gives rise to a map \( T_2 \) that is valid when \( 0 < \sin \theta_j \ll 1 \), i.e., when the simplification \( \sin \theta_j = O(1) \) breaks down. This situation involves new (persistent) heteroclinic connections, \( P_0 \rightarrow MM_\pi \) and \( P_\pi \rightarrow MM_\pi \) (see Fig. 14), and hence requires a separate treatment. Cycles incorporating these connections exist at \( \alpha = \alpha_\ast \), at which \( W^u(MM_\pi) \) intersects \( W^s(SW) \), generating (among other possibilities) the heteroclinic cycles \( O \rightarrow (P_0, P_\pi) \rightarrow MM_\pi \rightarrow SW \rightarrow O \). We analyze the situations near both of these heteroclinic cycles simultaneously by defining

\[
\begin{align*}
T_{O \rightarrow P} : & \begin{pmatrix} a_1 \\ \epsilon \\ \Delta \end{pmatrix} \mapsto \begin{pmatrix} b_j^2 a_1 \\ \epsilon \\ \Delta \end{pmatrix}, \\
T_P : & \begin{pmatrix} a_1 \\ \epsilon \\ \Delta \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ \epsilon \alpha_2/\alpha_1 \\ \Delta \end{pmatrix}.
\end{align*}
\]

Here the last entry in (43) follows from the invariance of the \( y = 0 \) plane.

We now need to derive a local map for \( MM_\pi \) and to this end we define two additional Poincaré sections,

\[
\Sigma^1_{MM} : \{ (\tilde{y}, R, \varphi) | 0 \leq \tilde{y} \leq \epsilon, R = \epsilon \}, \quad \Sigma^2_{MM} : \{ (\tilde{y}, R, \varphi) | \tilde{y} = \epsilon, 0 \leq R \leq \epsilon \},
\]

where \( \tilde{y}, R, \varphi \) are normal coordinates centered at \( MM_\pi \) which satisfy

\[
\tilde{y} = \delta \tilde{y}, \quad \tilde{R} = \gamma R, \quad \tilde{\varphi} = \Omega.
\]

In the regime we are considering that the eigenvalue \( \delta \) is positive, while \( \gamma \), the real part of the complex eigenvalues, is negative. With the time of flight given by

\[
\tau_{MM} = \frac{1}{\delta} \log \left( \frac{\epsilon}{\tilde{y}} \right),
\]

it is straightforward to obtain \( T_{MM} : \Sigma^1_{MM} \rightarrow \Sigma^2_{MM} \):

\[
T_{MM} : \begin{pmatrix} \tilde{y} \\ \epsilon \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ \epsilon \left( \frac{\gamma}{\tilde{y}} \right) \\ \varphi + \frac{\Omega}{\delta} \log \left( \frac{\epsilon}{\tilde{y}} \right) \end{pmatrix}.
\]

We also require two more global maps \( T_{P \rightarrow MM} : \Sigma^2_P \rightarrow \Sigma^1_{MM} \) and \( T_{MM \rightarrow SW} : \Sigma^2_{MM} \rightarrow \Sigma^1_{SW} \):

\[
T_{P \rightarrow MM} : \begin{pmatrix} \epsilon \\ \xi \\ \Delta \end{pmatrix} \mapsto \begin{pmatrix} \tilde{d}_j^2 \Delta \\ \epsilon \varphi_j + \hat{c}_j \xi + \hat{f}_j \Delta \end{pmatrix},
\]

\[
T_{MM \rightarrow SW} : \begin{pmatrix} \epsilon \\ \varphi \\ \Delta \end{pmatrix} \mapsto \begin{pmatrix} \tilde{d}_j^2 \Delta \\ \epsilon \varphi_j + \hat{c}_j \xi + \hat{f}_j \Delta \end{pmatrix}.
\]
This time the subscript $j$ distinguishes only the two cases $\theta_j = 0$ and $\theta_j = \pi$ and we have denoted $W^u(P_{\theta_j}) \cap \Sigma^1_{\text{MM}}$ by $(\tilde{y}, \tilde{R}, \tilde{\varphi}) = (0, \epsilon, \varphi_a)$ and $W^u(MM_\pi) \cap W^s(SW)$ which occurs at $\mu = 0$ by $(\tilde{y}, z, \tilde{\theta}) = (\epsilon, 0, \theta_a)$. The rescaled bifurcation parameter $\mu$ is a function of $\alpha$ and satisfies $\mu(\alpha_a) = 0$. For the sake of simplicity possible $\alpha$-dependence of the remaining coefficients is ignored; this does not affect the nature of our conclusions.

From the composition $T_{\text{MM}} \circ T_{\text{SW}} : T_{\text{MM}} \circ T_{\text{SW}}$ we get a map $T_2 : \Sigma^1_{\Omega} \to \Sigma^1_{\Omega}$ whose fixed points satisfy equations that are formally identical to the previous case.

\[
T_{\text{MM}} \to \text{SW} : \begin{pmatrix} \epsilon \\ \tilde{R} \\ \tilde{\varphi} \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ \tilde{g} \tilde{R} \cos(\varphi + \varphi_a) + \mu \\ \tilde{\theta}_a + \tilde{h} \tilde{R} \cos(\varphi + \varphi_b) \end{pmatrix},
\]

(48)

Here $Z(r, \Delta)$ again represents a coordinate transverse to $W^s(SW)$ and is required to be positive, but this time

\[
Z(r, \Delta) = \tilde{q}_j \Delta^{-\gamma/\delta} \cos \left( \tilde{\varphi}_j + \tilde{\alpha}_j + \tilde{f}_j \Delta - \frac{\Omega}{\delta} \log \Delta \right) + \mu.
\]

(51)

Moreover, $\Psi(r, \Delta)$ is the angle on $\Sigma^2_{\text{SW}}$ relative to $\theta_0$ and we must again define it mod $2\pi$. Thus

\[
\Psi(r, \Delta) - 2\pi n = \tilde{\theta}_a - \theta_0 + \tilde{u}_j \Delta^{-\gamma/\delta} \cos \left( \tilde{\varphi}_j + \tilde{\alpha}_j + \tilde{f}_j \Delta - \frac{\Omega}{\delta} \log \Delta \right) + \frac{\omega}{\rho} \log \left( \frac{Z}{\epsilon} \right),
\]

(52)

where

\[
\tilde{q}_j = \tilde{g} \epsilon^{1+\gamma/\delta} \tilde{d}_j^{-2\gamma/\delta}, \quad \tilde{\varphi}_j = \varphi_j + \frac{\Omega}{\delta} \log \left( \frac{\epsilon}{\tilde{d}_j} \right), \quad \tilde{\alpha}_j = \tilde{f}_j \epsilon^{1-\gamma/\delta} \tilde{d}_j^{-2\gamma/\delta},
\]

(53)

and $n$ again measures the number of SW oscillations. To solve Eqs. (49) and (50) we note first that $Z^{-\lambda/\rho}$ must be a higher order quantity (in $r$ and $\Delta$). Thus to leading order $\Psi = r/k$. It follows from Eqs. (50) and (52) that

\[
2\pi n + \beta - \frac{r}{k} + \tilde{u}_j \Delta^{-\gamma/\delta} \cos \left( \tilde{\varphi}_j + \tilde{\alpha}_j + \tilde{f}_j \Delta - \frac{\Omega}{\delta} \log \Delta \right) - \frac{\omega}{\lambda} \log (r \Delta) = 0,
\]

(54)

where

\[
\beta = \tilde{\theta}_a - \theta_0 - \frac{\omega}{\rho} \log \epsilon + \frac{2\omega}{\lambda} \log \tilde{r}.
\]

(55)

For small $r$ and $\Delta$ this equation immediately yields

\[
(r \Delta)_n = \tilde{c} \epsilon^{2\pi (\beta/\omega)n},
\]

(56)

where $\tilde{c} \equiv \epsilon^{\beta \lambda/\omega}$. Eq. (56) predicts an inverse relationship between $r$ and $\Delta$, i.e., an orbit (with given $n$) that comes close to the invariant plane $y = 0$ must at the same time move away from the plane $a_1 = 0$. This behavior is observed in our computations. Note that Eq. (56) is only valid when both $r$ and $\Delta$ are small and that the smallest
value of $\Delta$ is reached when $r$ is of order 1 (in units of $\epsilon$). At this point which we denote by $\Delta = \Delta_{\text{min}}$ the orbit is effectively released from the O and P states, and has been ‘captured’ by MM$_n$. Thus we define

$$\Delta_{\text{min}} = \epsilon^{-1} e^{2\pi \rho/\alpha}.$$  \hfill (57)

Using (56) in (50) then allows us to write down a single equation for $\Delta$:

$$\hat{\mu}_n = A_j \Delta^{-\gamma/\beta} \cos \left( \hat{\phi}_j + \hat{\delta}_j \left( \frac{\Delta_{\text{min}}(\theta)}{\Delta} \right)^\nu + \hat{\phi}_j \Delta - \frac{\Omega}{\delta} \log \Delta \right) + \hat{\mu},$$  \hfill (58)

where

$$A_j = \hat{q}_j (\hat{\epsilon}^{-2})^{\rho/\lambda}, \quad \hat{\mu} = \mu (\hat{\epsilon}^{-2})^{\rho/\lambda}, \quad \hat{\mu}_n = e^{-2\pi \rho/\alpha \Delta}.$$  \hfill (59)

The right-hand side of this equation must be positive (since $Z$ must be positive). For each value of $\hat{\mu}$ the resulting solutions determine the values of $\Delta$ corresponding to period-1 orbits, as shown (for given $n$) in Fig. 17. The figure also shows that as $\hat{\mu}$ passes through $\hat{\mu}_n$ the period-1 solutions go through a cascade of saddle-node bifurcations centered on $\hat{\mu}_n$ at which pairs of additional solutions appear. On an isola these saddle-nodes are located along the almost vertical boundaries of the twist region, where the map $T_2$ is valid. The fact that the map is valid only for $\Delta_{\text{min}} \lesssim \Delta \lesssim \epsilon$ suggests that for finite $n$ the cascade will be incomplete since $\Delta$ cannot become arbitrarily small. In fact $\Delta_{\text{min}} = 0$ only for $n = \infty$; in this case there is, at $\hat{\mu} = 0$, an infinite number of exact heteroclinic cycles of the type $O \rightarrow P_{\theta_k} \rightarrow SW \rightarrow O$ in addition to the cycles already mentioned. All of these cycles have $\sin \theta_k \ll 1$ and pass close to MM$_n$. For finite $n$ the accumulation points of these (incomplete) cascades occur at $\hat{\mu} = \hat{\mu}_n$. Consequently the accumulation points for successive $n$ form a geometric progression

$$\frac{\hat{\mu}_{n+1}}{\hat{\mu}_n} = e^{-2\pi \rho/\alpha \Delta},$$  \hfill (60)

this progression accumulates on $\hat{\mu} = 0$. Thus only on the ‘infinite period’ isola will there be an infinite number of saddle-node bifurcations along the twist portion. Fig. 18 provides a closer look at the twist portion of the higher period isolas for case B and clearly shows the geometric scaling described by (60) which in this case predicts a ratio of approximately 0.9, in reasonable agreement with the value $\approx 0.86$ obtained directly from Fig. 18.

One can obtain an idea of the number of saddle-node bifurcations present in the twist region from Fig. 17. If we denote successive saddle-nodes occurring (in order of decreasing $\Delta$) on alternating sides of a partial cascade centered at $\hat{\mu}_n$ by $\hat{\mu}_{n,k}$, $k = 1, \ldots, k_{\text{max}}(\Delta_{\text{min}})$, then Eq. (58) predicts that, for small $\Delta$,
Fig. 18. Twist portion of higher period isolas for case B.

\[
\frac{\hat{\mu}_{n,k+1} - \hat{\mu}_n}{\hat{\mu}_n - \hat{\mu}_{n,k}} = e^{\pi(\gamma/\Omega)}.
\] (61)

This equation yields the approximate values 0.04, 0.018, \(10^{-5}\), 0.33 and 0.3 for cases A, B, C, D and E, respectively. Although these numbers are only estimates since they depend on \(\alpha\) (i.e., on \(n\)) and the asymptotic analysis is not strictly valid in cases C–E they do provide a qualitative explanation for the different appearance of the twist region in the various cases. Specifically, cases A and B exhibit a small number of (visible) saddle-nodes, C has almost none, whereas D and E have numerous and easily visible saddle-node bifurcations within the twist region. The results (60) and (61) indicate that the structure of the twist region is determined by an interesting conjunction of the local properties of SW and MM.

We can also use (56) in (24), (27) and (45) to derive an estimate for the period of the \(n\)th isola as a function of \(\Delta\) when \(\sin \theta \lesssim \epsilon\):

\[
\tau_n = \bar{\tau}_j + \frac{2\pi}{\omega} \mu \left(1 + \frac{\lambda}{\mu_3} + \frac{\mu_1 \lambda}{\sigma_1 \mu_3} + \left(\frac{1}{\mu_3} + \frac{\mu_1}{\sigma_1 \mu_3} - \frac{1}{\delta}\right) \log \Delta.\right)
\] (62)

Here \(\bar{\tau}_j\) is a constant while the second term agrees with the previous expression (42) for the case when \(\sin \theta \sim 1\). The third term shows that the period can change dramatically as \(\Delta\) decreases toward \(\Delta_{\min}\). This is due to the fact that the orbit is being released from the O and P states as it is captured by MM. Whether one gets an increase or decrease in period depends on the relative magnitude of the eigenvalues for these three fixed points. For the parameters studied in this paper the quantity multiplying \(\log \Delta\) is positive which means that the decrease in time spent near O and the P states is greater in magnitude than the increase in time spent near MM; consequently we see a dramatic decrease in period as the orbit approaches the invariant plane \(\gamma = 0\). The resulting drop in period is responsible for the twist regions of the isolas. Moreover, since the drop in period (i.e., the length of the twist region) is governed by \(\Delta_{\min}\) which decreases with increasing \(n\) the twist region gets progressively longer as one proceeds up the hierarchy of isolas, becoming arbitrarily long as \(n \to \infty\) (see, e.g., Fig. 2b).

To establish a connection between the regime where \(T_1\) is valid and the regime just considered (where \(T_2\) is valid), we focus on a given isola, and follow it with decreasing \(\alpha\) from the initial saddle-node bifurcation near the point where \(W^u(P_0)\) first becomes tangent to \(W^u(SW)\). After this the orbits ‘slide’ along the circle of P states following closely the exact heteroclinic cycles at \(\theta = \theta_j\). In the simplest situation \(j = 1, 2\) and the \(\theta_j\) move with \(\alpha\) in such a
way that one moves monotonically toward $\theta = 0$ while the other approaches $\theta = \pi$. The intersection of $W^u(P_\alpha)$ with $W^s(SW)$ would then consist of two points with angles $\theta_1$ and $\theta_2$ which begin to move apart after the initial saddle-node where they are of course equal. As $\alpha$ approaches $\alpha_*$ (and $\theta_1$ and $\theta_2$ approach $0$ and $\pi$, respectively), most of $W^u(P_\alpha)$ moves outside $W^s(SW)$ but there is a small part (where $\sin \theta$ is small) that comes close to $MM_\pi$ and is therefore also stretched out along $W^u(MM_\pi)$. At this point $\theta_1$ and $\theta_2$ must both have become nearly equal to $\theta_*$ mod $2\pi$; see Fig. 19. Assuming that they have moved around $W^s(SW)$ in opposite directions (this is not the only possibility but it is the one which leads to isolas) we can write $\theta_2 - \theta_1 \approx 2\pi$ or equivalently $\psi_2 \approx \psi_1 + 2\pi$, which in turn implies by virtue of (41) that

$$\frac{r_2}{r_1} \approx e^{-2\pi (\lambda/\omega)}.$$  \hfill (63)

Thus the effect is the same as if $n$ had increased by 1 in going from one solution branch on the isola to the other. This is in essence what has happened. The upper branch ($j = 2$) has gradually acquired an additional SW oscillation with respect to the lower one ($j = 1$). It is a matter of convention that we have chosen to represent this effect as a continuous change in the $\psi_j$ rather than allowing $n$ to jump abruptly at some point along the isola. This argument suggests that in passing to the regime where $T_2$ is valid we should assign an effective value of $n + 1$ to the upper branch ($\theta \approx \pi$) of the $n$th isola. We therefore rewrite Eq. (58) showing explicitly the two cases:

$$\hat{\mu}_n = A_1 \Delta^{-\gamma/\delta} \cos \left( \psi_1 + \hat{s}_1 \left( \frac{\Delta_{\min}(n)}{\Delta} \right)^\nu + \hat{f}_1 \Delta - \frac{\Omega}{\delta} \log \Delta \right) + \hat{\mu},$$  \hfill (64)

$$\hat{\mu}_{n+1} = A_2 \Delta^{-\gamma/\delta} \cos \left( \psi_2 + \hat{s}_2 \left( \frac{\Delta_{\min}(n+1)}{\Delta} \right)^\nu + \hat{f}_2 \Delta - \frac{\Omega}{\delta} \log \Delta \right) + \hat{\mu}.$$  \hfill (65)

It follows that the accumulation point of the sequence of saddle-node bifurcations arising from the upper part of isola $n$ in the twist region should coincide with the accumulation point for the saddle-node bifurcations arising from the lower branch of isola $n + 1$. This prediction is borne out by the numerical results for cases A and B (and the others as well); see Fig. 18.
7.3. Poincaré map 3: \( SW \rightarrow MM \rightarrow SW \)

Finally we consider the lowest portion of the twist region. In this region the orbits no longer come close to either \( O \) or \( P \), and are instead close to the heteroclinic cycle \( SW \rightarrow MM \rightarrow SW \) (which exists at \( \alpha = \alpha_s \)). This region is described by the map \( T_3 \). To construct this map we need the map \( T_{SW \rightarrow MM} \):

\[
T_{SW \rightarrow MM} : \begin{pmatrix} \frac{\dot{y}}{\epsilon} \\ \frac{\dot{\theta}}{\epsilon} \end{pmatrix} \mapsto \begin{pmatrix} B^2 \frac{\dot{y}}{\epsilon} \\ f(\theta, \dot{y}) \end{pmatrix}.
\]

(66)

Here \( f(\theta, \dot{y}) \) is an undetermined function and we have again utilized the invariance of the \( y = 0 \) plane. By taking the composition \( T_{MM \rightarrow SW} \circ T_{MM} \circ T_{SW \rightarrow MM} \circ T_{SW} \) we get a map \( T_3 : \Sigma_{SW}^1 \rightarrow \Sigma_{SW}^1 \) the fixed points of which satisfy

\[
z = G z^\eta \cos \left( \tilde{\beta}_a + F(\theta, z) + \frac{\Omega \lambda}{\rho \delta} \log z \right) + \mu,
\]

(67)

\[
\dot{\theta} = \dot{\theta}_s + H z^\eta \cos \left( \tilde{\beta}_b + F(\theta, z) + \frac{\Omega \lambda}{\rho \delta} \log z \right),
\]

(68)

where

\[
\eta = \frac{\lambda}{\rho \delta}, \quad \tilde{\beta}_a = \varphi_a - \frac{\Omega}{\delta} \log (B^2 e^{\lambda/\rho}), \quad G = \hat{\delta} B^{-2\gamma/\delta} e^{1-\eta}, \quad \tilde{\beta}_b = \varphi_b - \frac{\Omega}{\delta} \log (B^2 e^{\lambda/\rho}),
\]

\[
H = \hat{h} B^{-2\gamma/\delta} e^{1-\eta}, \quad F(\theta, z) = f \left( \theta - \frac{\omega}{\rho} \log \left( \frac{\epsilon}{z} \right), \epsilon \left( \frac{\epsilon}{z} \right)^{\lambda/\rho} \right).
\]

(69)

Using \( \dot{\theta} = \dot{\theta}_s + O(z^n) \) in (67) gives

\[
z = G z^\eta \cos \left( \tilde{\beta}_a + F(\dot{\theta}_s, z) + \frac{\Omega \lambda}{\rho \delta} \log z \right) + \mu.
\]

(70)

The nature of the solutions to this equation depends on whether \( \eta > 1 \) or \( \eta < 1 \). For the parameters we consider \( \eta \) is quite large (because \( \rho \) is small) and thus there are no solutions for \( \mu < 0 \) and one unique solution for \( \mu > 0 \). It is also easy to show that this solution is stable (for small \( z \)). The full story is complicated however by the presence of \( W^s(O) \). If a trajectory happens to cross \( \Sigma_{SW}^2 \) near \( W^s(O) \cap \Sigma_{SW}^2 \) then this orbit will come very near to \( O \) and hence to a \( P \) state as well. This is just the reverse of the process described previously with the map \( T_2 \) and it occurs whenever

\[
\dot{\theta}_s - \frac{\omega}{\rho} \log \left( \frac{\epsilon}{z} \right) = \dot{\theta}_0 - 2\pi n.
\]

(71)

This equation in combination with (70) quickly reproduces (60):

\[
\frac{\mu_{n+1}}{\mu_n} = e^{-2\pi(\rho/\omega)}.
\]

(72)

We can now give a relatively complete picture for a high \( n \) isola in case A or B. After the initial saddle-node bifurcation there are two branches of period-1 orbits which follow the heteroclinic cycles \( SW \rightarrow O \rightarrow P \rightarrow SW \) as \( \alpha \) is decreased. When \( \alpha \) gets close to \( \alpha_s \), a transition occurs as the orbits move towards the \( y = 0 \) invariant plane and are simultaneously released by the O and P states while being captured by MM. Associated with this process is a partial cascade of saddle-nodes and a dramatic drop in period near the value of \( \alpha \) corresponding to \( \tilde{\mu}_n \) for the
lower branch and \( \hat{\mu}_{n+1} \) for the upper branch. The solutions created in this partial cascade annihilate pairwise in a reverse cascade except for the ones with the smallest \( \Delta \) (the lowest period) which escape completely from the O and P states. The lowest part of the twist connects the two escaped orbits and allows the number of SW oscillations to change smoothly by 1 (recall that the upper branch gradually acquired an extra SW oscillation with respect to the lower one). This lower portion of the twist approximates the heteroclinic cycle \( \text{SW} \rightarrow \text{MM}_\pi \rightarrow \text{SW} \) and is necessarily stable.

8. General 1:n resonance

In this section we briefly discuss the situation for a general 1:n spatial resonance and show that one can expect to find complex dynamics similar to that found for the 1:3 resonance.

The representation of the symmetry group O(2) appropriate for a general 1:n mode interaction is

\[
T_\phi : (z_1, z_n) \mapsto (e^{i\phi} z_1, e^{in\phi} z_n), \quad R : (z_1, z_n) \mapsto (\bar{z}_1, \bar{z}_n),
\]

and the most general smooth vector field equivariant under (73) has the form

\[
\begin{align*}
\dot{z}_1 &= p_1(u, v, w)z_1 + q_1(u, v, w)\bar{z}_1^{n-1}z_n, \\
\dot{z}_n &= p_n(u, v, w)z_n + q_n(u, v, w)\bar{z}_1^n,
\end{align*}
\]

where the invariants \( u, v, \) and \( w \) are given by

\[
\begin{align*}
u &= |z_1|^2, \\
w &= z_1\bar{z}_n + \bar{z}_1z_n.
\end{align*}
\]

One can reduce (74) to a three-dimensional system on writing \( z_j = a_j e^{i\phi_j} \) and defining the \( T_\phi \) invariant phase \( \Theta = \phi_n - n\phi_1 \). Setting \( X = a_n \cos \Theta \) and \( Y = a_n \sin \Theta \), we obtain

\[
\begin{align*}
\dot{a}_1 &= p_1a_1 + qa_1^{n-1}X, \\
\dot{X} &= p_nX + qa_1^n + nqa_1Y^2a_1^{n-2}, \\
\dot{Y} &= p_nY - nqa_1XYa_1^{n-2}.
\end{align*}
\]

It follows that for \( n > 3 \) the situation is essentially identical to the one analyzed in this paper. In particular, there is a circle of pure modes defined by \( a_1 = 0 \) and \( p_n = 0 \) which is not due to the symmetry \( T_\phi \) but to a degeneracy of the eigenvalues in the four-dimensional system. Thus for \( n > 3 \) we might expect to see the same types of global bifurcations as observed for \( n = 3 \). We have verified this conjecture for \( n = 4, \, n = 5 \) and \( n = 7 \). Fig. 20 illustrates the results for \( n = 5 \), and \( q_1 = 1, \, q_5 = -1, \, p_1 = -0.16u + 1.63v, \) and \( p_5 = -2.16u - 0.2v \). The bifurcation diagram is obtained by traversing a clockwise path in the \((\mu_1, \mu_5)\) plane defined by \((\mu_1, \mu_5) = (\cos \alpha, \sin \alpha)\), as before. Fig. 20b shows that in this case there is just one continuous solution branch rather than a stack of isolas, and that in contrast to cases A–E the twist region exhibits an abrupt increase in period, a possibility identified in Section 7.

In contrast, when \( n = 2 \), the degeneracy of the pure mode eigenvalues is lifted and the circle of pure mode fixed points is therefore absent in Eqs. (76). One might suppose therefore that this case, like the corresponding 1:1 resonance with \( O(2) \times Z_2 \) symmetry [21], would behave quite differently. Refs. [10,11,21] show that this is indeed the case. Nonetheless, it is still possible to choose coefficients such that the resulting equations exhibit the type of dynamics we have described here, although the details of the transitions are necessarily different [22].

9. Discussion

In this paper we described in some detail the unexpected complex dynamics that are present in the unfolding of a common type of mode interaction problem: the interaction of two steady modes with spatial wave numbers in the ratio 1:n in the presence of O(2) symmetry. Problems of this type have a variety of applications depending
on the origin of the O(2) symmetry. This symmetry can come about through the imposition of periodic boundary conditions on a problem posed on the real line or from the circular geometry of the domain. The former problem arises, for example, in the context of wavelength selection in Rayleigh–Bénard convection with midplane reflection symmetry, and it was in this context that the 1:3 resonance was first studied [9]. The latter is typified by experiments on cellular flames produced by a circular porous plug burner [23]. We have shown that the resulting mode interaction problem is extremely rich, and identified several different types of dynamics loosely related to the Shil’nikov–Hopf bifurcation. We also identified a number of distinct heteroclinic cycles, some of which were structurally stable and others structurally unstable, and explored the dynamics associated with them. Broadly speaking our results are related to those of Gaspard and Wang [24], Zimmermann et al. [18], and Champneys and Rodríguez-Luis [20]. The former identified for the first time the stacks of isolas that are associated with (homoclinic) connections to periodic orbits, while the latter two explored the mechanism by which such stacks of isolas form by analyzing degenerate forms of the (homoclinic) Shil’nikov–Hopf bifurcation. The present problem is significantly more complex because of the presence of heteroclinic cycles involving both periodic orbits and circles of equilibria. The dynamics described here, although complex, may be observable in partial differential equations such as those describing Rayleigh–Bénard convection at low Prandtl numbers [13].

The richness of this problem may perhaps be surprising since the general properties of steady-state mode interactions have been known for a long time. However, Dangelmayr [3] was the first to point out that the 1:n interactions differ fundamentally from the more general m:n (1 < m < n) mode interactions because of the absence of a branch of pure modes with wave number 1. Armbruster et al. [10] and Proctor and Jones [11] next noted that the 1:2 resonance was special because it admits attracting structurally stable heteroclinic cycles. Related cycles were also found in the 1:1 resonance [21] when the interacting modes have opposite parity. Rather different (and structurally unstable) cycles are present in the 1:1 resonance with algebraic multiplicity 2 but geometric multiplicity 1 (the so-called Takens–Bogdanov bifurcation). These cycles, identified by Dangelmayr and Knobloch [19], are also responsible for complex dynamics [25]. Interactions of this type were of great interest to John David Crawford who studied them extensively in the context of the Faraday system both in circular and square domains [26–28]. His work told us much about the role of hidden translation and rotation symmetries and the beautiful differences in the
properties of mode interactions in square domains and in nonsquare domains with D₄ symmetry [29,30]. He would have been amused to see that much still remains to be discovered even in a problem like the 1:3 resonance that, on the surface, seems very simple.

Acknowledgements

This work was supported in part by the National Science Foundation under Grant No. DMS-9703684.

References