Time-Modulated Oscillatory Convection

Hermann Riecke and John David Crawford

Institute for Nonlinear Science, University of California at San Diego, La Jolla, California 92093

Edgar Knobloch

Physics Department, University of California, Berkeley, California 94720

(Received 25 April 1988)

We investigate the effect of temporal modulation on the spatial patterns produced by a spatially extended system undergoing a Hopf bifurcation. It is shown that physically this modulation can stabilize standing waves, which would otherwise be stable to traveling waves. Mathematically a codimension-3 point is naturally introduced which encompasses a Takens-Bogdanov codimension-2 point with vanishing Hopf frequency. Experimentally this setup is attractive since the three relevant parameters are easily accessible and the steady bifurcation can be made forward as well as backward near a codimension-2 point.

PACS numbers: 47.25.Qx, 03.40.Kf, 47.35.+i

Symmetry plays an important role in the discussion of pattern forming systems since the appearance of a pattern is equivalent to breaking some symmetry of the system which was fully displayed in the basic state. These systems can, in general, produce a variety of patterns with different symmetries. The processes responsible for pattern selection close to threshold are strongly influenced by small symmetry-breaking perturbations.

The question we address here is how perturbing a time-translational symmetry affects the emerging spatial patterns if the form of the unperturbed patterns involves both, temporal and spatial, aspects of the basic symmetry. An example is provided by the traveling waves found in recent convection experiments on a binary fluid mixture in an annulus.\(^1\) With such periodic boundary conditions the bifurcation to time-dependent convection produces both traveling and standing waves.\(^2\)

The standing wave has only a spatial reflection symmetry, but the traveling wave is invariant under spatial translation followed by time evolution for an appropriate length of time. In the experiments, the standing waves are unstable to the traveling waves which are observed. We now imagine that the temporal symmetry is broken by external homogeneous modulation, providing an experimentally interesting realization of the phenomena described below.

In recent years several authors have considered the influence of temporal modulation on pattern formation resulting from a steady-state bifurcation.\(^3\) In this Letter we consider Hopf bifurcation in spatially extended systems and study the effect of modulating the Rayleigh number \(\mathcal{R}\) with a frequency \(\omega_e\) that is resonant with the Hopf frequency \(\omega_H\) of the unperturbed waves. This modulation affects the traveling and standing waves differently because of their different symmetries. In particular, it is possible to stabilize the standing waves and excite them for Rayleigh numbers at which the traveling waves do not exist. In addition, a codimension-2 Takens-Bogdanov bifurcation, at which the Hopf frequency vanishes, is obtained quite naturally in this system by our adjusting \(\mathcal{R}, \omega_e\), and the modulation amplitude. All of these parameters are easily adjustable.

To formulate the problem mathematically, we consider a Hopf bifurcation in a homogeneous extended system with periodic spatial boundary conditions and time-periodic modulation. Details of the calculations will be presented elsewhere.\(^4\) The equations of motion and the basic state are assumed to be invariant with respect to spatial translations \(z \rightarrow z + d\) and spatial reflections \(z \rightarrow -z\); hence, the bifurcation occurs in the presence of \(O(2)\) symmetry. For small modulation \(a(t)\) and close to the bifurcation point \(\mathcal{R}_c\) (\(a = 0\)) the dynamics of the system can be reduced to a center manifold \((u_1, u_2, a)\), where the \(u_i\) give the complex amplitudes for the linear modes

\[
\psi = u_1(t)e^{iq_1}\Phi_1(r) + u_2(t)e^{iq_2}\Phi_2(r) + \text{c.c.}
\]

deep into the eigenvalues \(+i\omega_H\) and \(-i\omega_H\), respectively. Here \(\psi\) describes the state of the system, e.g., for binary convection it gives the velocity and temperature of the fluid and the concentration, and \(r\) denotes all other independent variables. The form of \(\psi\) determines how the \(O(2)\) symmetry acts on the center manifold variables; letting \(T\) denote translations by \(d\) and spatial reflections by \(K\), we have \(T(u_1, u_2, a) = (e^{i\varphi}u_1, e^{i\varphi}u_2, a)\) and \(K(u_1, u_2, a) = (a, u_2^*, u_1^*, a)\).

The equations for \(u_1\), \(u_2\), and \(a\) will be symmetric with respect to this action which implies they have the form

\[
\begin{align*}
\partial_t u_1 &= g_1u_1 + g_2u_2, \\
\partial_t u_2 &= g_2^*u_1 + g_1^*u_2, \\
\partial_t a &= i\omega_e a.
\end{align*}
\]

Here the \(g_i\) are complex functions of \(a\) and the invariants \(|u_1|^2 + |u_2|^2, u_1u_1^*,\) and \(u_1^*u_2\). Depending on the ratio \(\gamma = \omega_H/\omega_e\) these equations can be simplified further by...
nonlinear coordinate transformations. In this discussion we restrict ourselves to the strong resonance $\gamma = \frac{1}{4}$. For $\eta = u_1 \exp(-i\omega_0 t) + \cdots$, $\zeta = 2\exp(+i\omega_0 t) + \cdots$, and $\xi = a\exp(-i\omega_0 t)$ (with the ellipses denoting higher-order terms), one then obtains to cubic order

\begin{align}
\partial_t \eta &= a\eta + b\xi + c\eta(|\eta|^2 + |\zeta|^2) + g\eta|\xi|^2 + \cdots, \\
\partial_t \zeta &= a^*\zeta + b\eta + c^*\zeta(|\eta|^2 + |\zeta|^2) + g^*|\xi|^2 + \cdots.
\end{align}

(2a)

(2b)

The coefficient $b$ has been made real and positive by a suitable choice of phases; the other coefficients remain complex. The frequency ratio $\gamma = 1$ yields equations of the same form since the $O(2)$ symmetry suppresses constant terms. Thus these two resonances differ only in the $\zeta$ dependence of $b$: $b \propto \zeta^2$.

The external control parameters are $a = a_R + ia_I$, and $b$, where $a_R$ is essentially the Rayleigh number, $a_I$ represents the deviation from exact resonance (detuning), and $b$ measures the modulation amplitude: $\mathcal{R} = \mathcal{R}_c + a_R A + b c \cos \omega_0 t$, and $\omega_0 = (\omega_1 + a_R B - a_I)/\gamma$ with $A$, $B$, and $C$ being $O(1)$ quantities. Thus, we are quite naturally led to consider the parameter values close to the codimension-3 points $a_R = 0$, $a_I = 0$, and $b = 0$ which act as an “organizing center” for this problem.

The linear stability analysis of (2) around the trivial state $\eta = \zeta = 0$ shows a Hopf bifurcation (H) at $a_R = 0$ for $b^2 < a_i^2$ and a steady bifurcation (S) at $a_R^2 = b^2 - a_i^2$ for $b^2 > a_i^2$. Therefore, at $a_R = 0$ and $b^2 = a_i^2$, we reach a Takens-Bogdanov codimension-2 bifurcation (TB). Mathematically this point is identical to the one obtained in binary mixture convection by our varying the separation ratio to go from oscillatory to steady convection.

It is useful to introduce $\eta = x \exp(i\varphi_1)$, $\zeta = y \exp(i\varphi_2)$, $\chi = \varphi_1 - \varphi_2$, and $\phi = \varphi_1 + \varphi_2$. Under spatial transitions the phase $\phi$ varies uniformly, whereas the phase $\chi$ is invariant. For these variables one obtains

\begin{align}
\partial_t x &= a_R x + b y \cos \chi + c_R x (x^2 + y^2) + g_R x y^2, \\
\partial_t y &= b x \cos \chi + a_R y + c_R y (x^2 + y^2) + g_R x y, \\
\partial_t \chi &= 2a_I + n_2 (x^2 + y^2) - b \sin \chi (x^2 + y^2)/xy,
\end{align}

with $n_2 = 2c_i + g_i$, and a decoupled equation for the spatial phase $\phi$. It is important to note that because of the modulation ($b > 0$) the temporal phase $\chi$ and the amplitudes $x$ and $y$ do not decouple. This is in contrast to the effect of perturbing the spatial translation symmetry in which case $\phi$ couples to the amplitudes and $\chi$ does not. In terms of the amplitudes and phases the linear mode (1) now reads

$$
\Psi = x(t) e^{i[(\phi + \chi)/2 + \omega_0 t + q_2\phi_1 + y(t) e^{i[(\phi - \chi)/2 - \omega_0 t + q_2\phi_2 + c.c.]}}.
$$

There are two types of stationary solutions ($\partial_t x = 0$, $\partial_t y = 0, \partial_t \chi = 0$): (a) $x \neq y, \partial_t \phi \neq 0$, referred to as traveling waves (TW); (b) $x = y, \partial_t \phi = 0$, which are steady states within the amplitude equations (2) and corresponding to standing waves (SS) in the original system. In the TW case one finds both left ($l$) and right ($r$) traveling solutions,

$$
\begin{align}
\chi^{2r} &= -a_R [1 \pm (1 - 4\Delta^2)^{1/2}] / 2c_R, \\
\chi^{2l} &= x^{2l}_r,
\end{align}
$$

which exist as long as $\Delta^2 = b^2 / (a_R^2 + 4\Delta^2) < 1$ with $\Omega = a_R c_R - a_R n_2 / 2$. With modulation ($b > 0$) pure right-($x = 0$) or left-traveling ($y = 0$) waves do not exist. Instead, the modulation mixes the two and this mixing increases until $\Delta^2 = \frac{1}{4}$, where both waves merge with the steady state $x = y$. Unlike pure traveling waves, these mixed TW have two independent frequencies $\partial_t \phi \pm \gamma \omega_0$, and topologically correspond to a 2-torus. The translation symmetry suppresses the mode-locking behavior which would generically be expected for such a 2-torus.

The SS solutions are given by

$$
\chi^2 = -\frac{1 - N^2 (a_R^2 + a_i^2 - b^2)/M^2}{N^2},
$$

with $M = a_R n_1 + a_R n_2, N^2 = n_2^2 + n_1^2$, and $n_R = 2c_R + g_R$. These solutions represent waves with frequency $\gamma \omega_0$, which are phase locked to the external modulation.

We now discuss the phase diagrams for these two kinds of states. We choose $c_R < 0$ and $g_R < 0$ so that both traveling and standing waves appear supercritically in the unmodulated system with the standing waves (SW) being unstable with respect to the stable traveling waves. The bifurcation diagram for this case ($b = 0$) is sketched in Fig. 1.

With modulation one has to distinguish between essentially two cases: $a,n_1 > 0$ and $a,n_1 < 0$. For $a,n_1 > 0$ (Fig. 2), traveling waves exist to the right of line H and below M: At H they branch supercritically from the trivial state, whereas at M they merge with SS as discussed above. The SS is stimulated by the modulation and appears supercritically already for $a_R < 0$ along S. Between S and SN two SS exist, which undergo a saddle-node bifurcation at SN. One SS undergoes a Hopf bifurcation along SW which is always preceded by a steady bifurcation to TW at M. The inset of Fig. 2 shows the bifurcation diagram obtained by our increasing $a_R$ along the dotted line.

For opposite detuning, $a,n_1 < 0$ (Fig. 3 and 4), the

FIG. 1. Bifurcation diagram without modulation for $c_R < 0, g_R < 0$. 1943
The analysis of Dangelmayr and Knobloch\textsuperscript{7} implies that close to TB, in addition to the SS solutions and the TW, there exist also SW and MW. In the present context these solutions correspond to slowly modulated standing waves and to modulated waves with three frequencies, respectively. For these two solutions, phase locking to the modulation is not excluded by symmetry and can therefore be expected generically. The SW and MW should therefore exhibit one and two independent frequencies, respectively. For our present choice of parameters ($c_R$ and $g_R$ negative) these solutions are not stable for $a_R n_i > 0$. For $a_R n_i < 0$, however, one finds stable MW solutions and—for $g_R / c_R < 3$—also stable SW. Along the dotted line in Fig. 3 one therefore obtains the bifurcation diagram shown schematically in Fig. 4, which implies the following hysteretic succession of events when changing the Rayleigh number $R$ ($a_i$ and $b$ being held fixed): increasing $a_R$ from negative values the conduction state will be stable up to $a_R = 0$, where TW appear. They undergo a Hopf bifurcation to stable MW at D. Upon further increasing $a_R$, the MW merge with the SW, which in turn end on the unstable SS via a heteroclinic orbit. The system will then jump to the stable SS. Now $a_R$ can be reduced all the way to negative values before losing the SS in a saddle-node bifurcation. Note that this hysteresis loop—excepting the SW—should be generically obtained by modulating any $O(2)$-symmetric supercritical Hopf bifurcation that leads to stable traveling waves.

Gambaudo\textsuperscript{10} shows that the heteroclinic orbit formed when the SW merge with the SS can be split by higher-order corrections. This creates transverse intersections in the stable-unstable manifolds of the SS solutions with the associated chaotic dynamics. Since SW are stable for $a_R n_i < 0$ if $g_R / c_R < 3$ some aspects of this chaos might in fact be observable close to the transition from SW to the unstable branch of SS. A more detailed analysis is deferred to subsequent work.

In conclusion, time-periodic modulation of a Hopf bifurcation in a spatially extended system with periodic boundary conditions has been investigated for the first
time. It has been shown that such a modulation at resonance can stabilize otherwise unstable standing waves. This result suggests that more generally it might be possible to stabilize unstable periodic states by external modulation at resonance. In addition, a codimension-2 point, where a Hopf bifurcation coalesces with a steady bifurcation, occurs when the modulation strength and the detuning balance. Close to this point stable modulated waves are predicted to bifurcate from (mixed) traveling waves. These results are generic for the modulation of a system which exhibits a Hopf bifurcation to traveling waves in the absence of modulation (e.g., binary mixtures, oscillatory instability in pure fluid convection, nematic liquid crystals). Experimentally this point should be easily accessible as the relevant external parameters besides the Rayleigh number are the modulation amplitude and its frequency. Since this codimension-2 point is embedded naturally in a singular point of codimension-3, one should expect additional phenomena, which cannot occur close to the codimension-2 point. In particular, it will be interesting to compare the dynamics with that obtained for the modulated Hopf bifurcation in a nonextended system where no continuous symmetry is present.

While completing these calculations we were informed that Walgraef and Coullet independently have obtained very similar results. H.R. wishes to acknowledge his discussion with D. Walgraef. This work has been supported by Air Force Grant No. AFOSR F49620-87-C-0117 and the Association for Computing Machinery program of the U.S. Defense Advanced Research Projects Agency.

4. H. Riecke, J. D. Crawford, and E. Knobloch, to be published.
5. For binary mixtures in porous media [H. Brand and V. Steinberg, Phys. Lett. 93A, 333 (1983)] we obtain $A = 4(1 + \omega L)/(1 + \psi^2)$, $B = \omega_0/2\pi$, $C = 4\Lambda^2/(1 + B^2)$ with $\Lambda$ being the Lewis number and $\psi$ the separation ratio ($\psi = \frac{1}{2}$, free-slip and fixed-concentration boundary conditions).
11. I. Rehberg and V. Steinberg, to be published.
12. Following our suggestions, I. Rehberg, S. Rasenat, J. Fineberg, M. de la Torre Juarez, and V. Steinberg, to be published, have observed standing waves in nematic liquid crystals.