Stellar dynamos are governed by non-linear partial differential equations (PDEs) which admit solutions with dipole, quadrupole or mixed symmetry (i.e. with different parities). These PDEs possess periodic solutions that describe magnetic cycles, and numerical studies reveal two different types of modulation. For modulations of Type 1 there are parity changes without significant changes of amplitude, while for Type 2 there are amplitude changes without significant changes in parity. In stars like the Sun, cyclic magnetic activity is interrupted by grand minima that correspond to Type 2 modulation. Although the Sun’s magnetic field has maintained dipole symmetry for almost 300 yr, there was a significant parity change at the end of the Maunder Minimum. We infer that the solar field may have flipped from dipole to quadrupole polarity (and back) after deep minima in the past and may do so again in the future. Other stars, with different masses or rotation rates, may exhibit cyclic activity with dipole, quadrupole or mixed parity. The origins of such behaviour can be understood by relating the PDE results to solutions of appropriate low-order systems of ordinary differential equations (ODEs). Type 1 modulation is reproduced in a fourth-order system while Type 2 modulation occurs in a third-order system. Here we construct a new sixth-order system that describes both types of modulation and clarifies the interactions between symmetry-breaking and modulation of activity. Solutions of these non-linear ODEs reproduce the qualitative behaviour found for the PDEs, including flipping of polarity after a prolonged grand minimum. Thus we can be confident that these patterns of behaviour are robust, and will apply to stars that are similar to the Sun.

Key words: chaos – MHD – Sun: magnetic fields – sunspots – stars: magnetic fields.

1 INTRODUCTION

The magnetic activity of a lower main-sequence star depends upon its rotation rate. As the star ages, it loses angular momentum, owing to magnetic braking, and so becomes less active. Magnetic cycles are found in middle-aged stars like the Sun, which are relatively inactive (e.g. Weiss 1994; Wilson 1994; Pallavicini 1996). The cycles have periods of 7–15 yr and are modulated on a longer timescale. This modulation leads to intervals of reduced activity (grand minima). The Sun’s magnetic history can be traced back for the last 10 000 yr, and the proxy record derived from abundances of cosmogenic isotopes such as $^{14}$C and $^{10}$Be demonstrates that the Maunder Minimum of the seventeenth century – when sunspots virtually disappeared – was preceded by numerous similar episodes. The last three of these grand minima – the Spörer Minimum (1415–1535), the Maunder Minimum (1645–1715) and the abortive Dalton Minimum (1795–1825) – were separated by intervals of 100–200 yr (cf. Beer, Tobias & Weiss 1998) and the $^{14}$C record displays a characteristic time-scale of about 200 yr; however, the record is aperiodic and the intervals between successive grand minima show considerable variation about this mean value. There is also evidence that up to 30 per cent of solar-type stars may be undergoing similar grand minima at any time: indeed, one star (HD3651) seems to have been caught in the act of entering a grand minimum.

The spatial structure of these magnetic fields can only be determined for the Sun. Since the end of the Maunder Minimum the incidence of sunspots has been symmetrical about the equator, with minor statistical fluctuations. Magnetic fields have only been measured since the beginning of this century and we now know that the azimuthally averaged toroidal field, which emerges in sunspots and active regions, is antisymmetric about the equator and reverses after each 11-year activity cycle. The corresponding poloidal field can be derived from a vector potential that is roughly symmetric about the equator; thus it has the symmetry of a dipole. At the end of the Maunder Minimum these symmetries were, however, broken (Ribes & Nesme-Ribes 1993). Between 1672 and 1712 all but three of the spots that were observed lay in the southern hemisphere; for
much of this period there were very few spots at all, and the first clear cycle that appeared (with its maximum in 1706) was almost entirely confined to one hemisphere; rough symmetry was only restored for the next cycle.

To explain the origin of these magnetic fields we must exploit the solar–stellar connection, and combine our detailed knowledge of systematic spatiotemporal behaviour on the Sun with information gleaned from other stars with different rotation rates. It is generally accepted that magnetic cycles in a star like the Sun are produced by a dynamo located at, or near, the base of its convection zone (see Weiss 1994, Weiss & Tobias 1997 and references therein). Strong toroidal fields can readily be generated in a region of weak convective overshoot, where helioseismology has revealed strong radial gradients in angular velocity within the Sun (the tachocline). These fields can be held down by the overlying turbulence, through the combined effects of flux expulsion and turbulent pumping by compressible convection (Tao, Proctor & Weiss 1998; Tobias et al. 1998), until instabilities driven by magnetic buoyancy allow flux ropes to escape into the convection zone, and then to emerge in active regions and form starspots. These stellar dynamos differ from the geodynamo (which maintains the same polarity for times much longer than the ohmic diffusion time for the Earth’s core) in that their magnetic fields reverse on a much shorter time-scale. Self-consistent non-linear calculations have confirmed that this oscillatory dynamo process works, but so far there have been no models that describe the behaviour of the solar cycle in as much detail as the massive computations that reproduce geomagnetic reversals (Glatzmaier & Roberts 1995). Most studies of stellar dynamos have instead relied on mean field dynamo theory. Although the conditions under which the mean field (or \( \alpha - \omega \)) dynamo equations can be justified do not apply in stars, those equations do capture the essential physics and they can be used to establish generic properties of stellar dynamos.

The measured modulation of the solar cycle is aperiodic and might have a stochastic or a deterministic cause. The available record is not long enough to distinguish between these two possibilities. Stochastic fluctuations in the turbulent convection zone could lead to on–off intermittency. Indeed, it has been shown that order-one fluctuations in the \( \alpha \) effect (which generates a reversed poloidal field from the toroidal field) can produce modulation similar to that which has been observed (Schmitt, Schüssler & Ferriz Mas 1996; Schüssler, Schmitt & Ferriz Mas 1997). We shall adopt the alternative hypothesis: that modulation is deterministic and apparently chaotic. On this assumption, weak stochastic perturbations allow the non-linear system to evolve on trajectories that shadow those on the chaotic attractor, and are qualitatively unimportant (unless the solutions enter deep minima).

Axisymmetric dynamos possess solutions with two different symmetries. If the toroidal field is \textit{antisymmetric} about the equator and the poloidal field is derived from a symmetric vector potential (as in the Sun at present) then the overall field is said to have \textit{dipole} symmetry; conversely, a field with a \textit{symmetric} toroidal component and an antisymmetric vector potential for the poloidal field is said to have \textit{quadrupole} symmetry. The symmetries of these pure modes may, however, be broken at a bifurcation. For instance, a periodic dipole (or quadrupole) solution may undergo a pitchfork bifurcation that leads to a periodic mixed mode solution (Jennings & Weiss 1991; Roald & Thomas 1997), or a Hopf bifurcation that leads to a quasiperiodic, modulated mixed mode solution.

The aim of this paper is to clarify the interactions between modulation and symmetry-breaking as generic features of non-linear stellar dynamos. Previous studies of modulation, which have been restricted to mean field dynamos, have revealed two distinct types of behaviour (Tobias 1997a, Weiss & Tobias 1997). The first is associated predominantly with symmetry changes, while the second corresponds to changes of amplitude without significant changes in symmetry. To distinguish between them, we define the parity

\[ P = \frac{E_Q - E_D}{E_Q + E_D}, \]

where \( E_O, E_D \) are the spatially averaged magnetic energies in the fields with quadrupole and dipole symmetry, respectively.

For modulations of Type 1, the parity varies considerably while the cycles undergo periods of reduced activity; quasiperiodic and chaotic modulations of this type have been found in several axisymmetric dynamo models, whether in spheres (e.g. Brandenburg et al. 1989a; Kitchatinov, Rüdiger & Küker 1994; Tavakol et al. 1995; Tworkowski et al. 1998) or tori (Brooke & Moss 1994, 1995), and has been related to grand minima (Brandenburg, Krause & Tuominen 1989b; Brandenburg et al. 1990). The same patterns occur not only in systems governed by partial differential equations (PDEs) but also in low-order models. Knobloch & Landsberg (1996, henceforth KL96) have investigated the behaviour of a fourth-order system of ordinary differential equations (ODEs) that describes dipole–quadrupole interactions in the weakly non-linear regime, and they find transitions from pure dipole or quadrupole modes to mixed parity states with Type 1 modulation.

Type 2 modulation is associated with grand minima and the parity \( P \) does not vary significantly (except when the total energy is very small). Tobias (1996) first demonstrated the existence of this type of modulation for PDEs by considering a Cartesian dynamo model with dipole symmetry imposed. The same type of modulation can also be found in low-order models: it was first demonstrated for a sixth-order system of ODEs representing a truncated model of one-dimensional dynamo waves (Weiss, Cattaneo & Jones 1984); subsequently, Tobias, Weiss & Kirk (1995, henceforth TWK95) showed that Type 2 modulation was reproduced by solutions of the normal form equations for a saddle-node/Hopf bifurcation. These third-order ODEs provide a robust minimal description of grand minima.

We shall present a detailed comparison between PDEs and ODEs. Parker (1993) introduced a simplified mean field dynamo model in Cartesian geometry, in which the shear (the \( \omega \) effect) was confined to the lower half-space (\( z < 0 \)), while the cycles undergo periods of reduced activity; quasiperiodic waves along the interface at \( z = 0 \). Tobias (1997b) developed a non-linear version of this model and demonstrated the presence of quasiperiodically and chaotically modulated waves. He also considered behaviour in a bounded region, which allowed solutions with dipole and quadrupole symmetry to appear (Tobias 1997a). In Section 2 we give examples of solutions to these PDEs and demonstrate the presence of periodic dipole solutions (\( P = -1 \)) and periodic quadrupole solutions (\( P = 1 \)), as well as stable dipole solutions with modulations of Type 2. Such solutions may of course undergo symmetry-breaking bifurcations and we also include a mixed mode solution with a modulation of Type 1. In addition, we illustrate an example of a mixed mode solution with Type 2 modulation, in which the parity flips at a grand minimum (Beer et al. 1998).

Low-order models allow us to understand generic properties and to explain the origin of complicated patterns of behaviour (TWK95). Although normal form equations have no detailed
predictive power, their bifurcation structures are robust and their dynamical properties are shared by a wide class of non-linear systems. In order to construct an ODE model that describes modulation of both types we need to merge the fourth-order model of KL96, which describes dipole quadrupole interactions, with the third-order equations that represent grand minima in a system with fixed parity (TWK95). In Section 3 we develop an appropriate sixth-order system that contains the minimal number of terms with relevant symmetries. Non-linear results are then presented in Section 4. With carefully chosen coefficients we can demonstrate Type 2 modulation, through transitions from periodic to quasiperiodic chaotic modulated solutions, all with pure dipole symmetry. We also illustrate two different mixed mode solutions, with modulation first of Type 1 and then of Type 2. As in many non-linear systems, there may be more than one stable solution for the same parameter values, with sensitive dependence on initial conditions.

From a mathematical viewpoint, it is important to note that both the PDEs and the ODEs possess a pair of invariant subspaces containing solutions with either pure quadrupole or pure dipole symmetry \((P = \pm 1)\). For Type 2 modulation, trajectories often lie close to one of these subspaces and approach the origin (where the magnetic field \(B = 0\)) during deep grand minima. This facilitates flipping between one parity and another, which is found for the ODEs as well as for the PDEs. Related behaviour, also involving symmetries, invariant subspaces and homoclinicities, occurs in other systems too (e.g. Matthews et al. 1996; Ashwin & Rucklidge 1997).

Having established that the ODEs reproduce qualitative features of the non-linear solutions found for a highly idealized PDE model, we are in a position to claim that those features are generic and robust. In the final section, we therefore relate our results to observed magnetic fields on the present-day Sun. Then we go on to discuss their implications for the behaviour of the Sun in the past and future, and thus for the magnetic properties of other stars.

2 THE PDE MODEL

In this section we summarize the results of a non-linear mean-field dynamo model that (in contrast to those described in the Introduction) allows both Type 1 and Type 2 modulation to occur and that can be used to investigate the interaction between the two. Since this model has been studied in detail elsewhere (Tobias 1997a; Beer et al. 1998) only the main points are included here, so that they may be compared with the results of the low-order model in Section 4 and used to draw conclusions about the possible range of behaviour of solar and stellar dynamos.

This two-dimensional Cartesian model of a dynamo operating in the solar overshoot region is an extension of that proposed by Parker (1993). Here the region of strong radial shear below the convection zone (the tachocline) efficiently generates toroidal field whilst the cyclonic turbulence of the convection zone produces the poloidal component of the field via the \(\alpha\) effect. In the two-dimensional system, \(z\) corresponds to the radial coordinate \((-1 \leq z \leq 1)\) whilst \(x\) is the colatitude. The computational domain is chosen to have latitudinal extent \(0 \leq x \leq 2L\) so that \(x = 0, 2L\) represent the north and south poles, respectively, with the equator at \(x = L\). The \(\alpha\) effect is antisymmetric about the equator \((\alpha = \alpha_0 \cos(\pi x/2L))\) and the imposed flow (giving rise to the shear) is symmetric. Hence the dynamo profile (a product of the shear and the \(\alpha\) effect) is antisymmetric about the equator. Thus the solutions that arise via an instability of the trivial field-free state are restricted to have either dipole or quadrupole parity. Mixed mode solutions (i.e. solutions that are asymmetric with respect to the equator) may only occur as a result of a secondary instability.

Once a magnetic field is generated it acts back on the velocity via the non-linear Lorentz force. In this model only the macrodynamic back-reaction of the large-scale magnetic field in driving a large-scale velocity perturbation (the ‘Malkus–Proctor effect’) is considered. Although this is a natural non-linearity to include – since it corresponds to the ‘torsional oscillations’ that are observed at the solar photosphere – it does require the solution of an additional equation for the velocity perturbation, and has therefore not been studied in as much detail as simpler quenching mechanisms. There are of course other non-linear processes (e.g. microdynamic quenching mechanisms, magnetic buoyancy) that might be introduced. Although the relative importance of these different effects in the Sun is still poorly understood (see e.g. Tobias 1997b) it is clear that if the large-scale magnetic field at the base of the convection zone is sufficiently strong then a magnetically driven shear must result. Furthermore, if the magnetic Prandtl number \(\tau = \eta/\nu\) is small enough then the Malkus–Proctor effect naturally leads to Type 2 modulation as discussed in Section 1. We emphasize, however, that in this paper we are striving to develop a formulation that is able to describe modulation of cyclic activity independently of the saturation mechanism considered.

In mean field dynamo calculations, it is usual to examine the transitions in the form of the generated field as the driving in the system is increased. This driving is characterized by the non-dimensional dynamo number

\[
D = \frac{\alpha_0 \omega_0 L^3}{\eta^2},
\]

where \(\alpha_0\) and \(\omega_0\) are measures of the efficiency of the \(\alpha\) effect and the strength of the shear, respectively, \(L\) is a typical length-scale of the problem and \(\eta\) is the (turbulent) magnetic diffusivity. The simplest non-linear solutions found in this system are periodic dipole or quadrupole modes as shown in the butterfly diagrams in Fig. 1(a,b). These modes lose stability in secondary bifurcations to modulated solutions. For reasonably small dynamo numbers the preferred type of modulation is Type 1 and quasiperiodic mixed modes are found. Here, as in other PDE models, the interaction between dipole and quadrupole modes leads to modulation of the basic magnetic cycle (see Fig. 2a). The parity of solutions changes as energy is transferred between the dipole and quadrupole modes (see Fig. 2b) but there is no event that resembles a minimum in energy of the type found in the Sun. The modulation appears in a butterfly diagram as a shift in the phases of the oscillation in the northern and southern hemispheres (cf. Brandenburg et al. 1989a) as shown in Fig. 1(c). The Malkus–Proctor effect is of secondary importance: the fields are small and little energy is transferred to the velocity perturbation \((v)\), as shown clearly in the three-dimensional phase portrait displayed in Fig. 3(a). There the trajectory in the infinite-dimensional phase space of the system is projected on to the space spanned by \((E_0, E_1, \langle v^2 \rangle\)), where \(E_0\) is the spatially averaged energy in the dipole field, \(E_1\) is the averaged quadrupole energy and \(\langle v^2 \rangle\) is the averaged kinetic energy of the velocity perturbations. The solutions lie on a two-torus and the changes in kinetic energy can be seen to be small. The modulation is clearly the result of energy transfer between the dipole and quadrupole modes. Because the magnetic field has both dipole and quadrupole components, the Lorentz force generates velocity perturbations that are asymmetric with respect to the equator, as shown in Fig. 4(a).
Figure 1. Butterfly diagrams: grey-scale plots of the toroidal field $B$ at $z = 0.5$, as a function of colatitude $x$ and time $t$, for the PDE model described in Section 2. The panels show four different types of solution. (a) Periodic dipole ($D = 300$, $\tau = 0.1$), with $B$ antisymmetric about the equator ($x = L$); (b) periodic quadrupole ($D = 500$, $\tau = 0.1$), with $B$ symmetric; (c) quasiperiodic mixed mode ($D = 400$, $\tau = 0.1$), showing Type 1 modulation; (d) flipping between different parities ($D = 1100$, $\tau = 0.025$), with asymmetric $B$.

Figure 2. Type 1 modulation: a quasiperiodic mixed mode PDE ($D = -400$, $\tau = 0.1$) solution showing (a) the spatially averaged magnetic energy ($B^2$) as a function of time and (b) the parity $P$ versus $t$. The amplitude of the basic cycle is modulated and this modulation is associated with large changes in the parity of the solutions.
As $D$ is increased these mixed modes lose stability to pure dipole solutions. Then, for yet larger values of the dynamo number, Type 2 modulation occurs – that is, solutions are modulated as a result of a transfer of energy from the magnetic field to the velocity perturbations via the Malkus–Proctor effect. This type of modulation is very different from that of Type 1 discussed earlier, as can be seen on comparing the three-dimensional phase portrait in Fig. 3(b) with that in Fig. 3(a). The dipole energy is modulated even though the quadrupole energy stays fixed at zero. Moreover, the velocity perturbations driven by this field are symmetric about the equator (as shown in Fig. 4b). Only if the magnetic field is mixed can an asymmetric part of the velocity perturbation be driven. The modulation gets stronger as $D$ is increased further, until the modulated solutions exhibit grand minima (i.e. the trajectories come very close to the invariant subspace $B^2 = 0$). During deep grand minima the solutions can be seen to be mixed: the (considerably) weaker quadrupole component becomes visible when the dipole field is small enough.

For solutions where the primary source of modulation is the Malkus–Proctor effect, energy may be transferred between the dipole and quadrupole modes, but usually only via the velocity perturbations. The velocity perturbations store the energy from the field (in this case the dipole mode) and return the energy to either the dipole or quadrupole mode on a time-scale controlled by the magnetic Prandtl number $\tau$. An extreme example of this is the ‘flipping’ described by Beer et al. (1998), which is illustrated in the butterfly diagram of Fig. 1(d). Here the grand minima may act as a switch between the two modes, so that solutions may enter a minimum as a dipole yet emerge predominantly quadrupolar. This is a clear example of how energy can be transferred between the modes via the velocity perturbation.

The three-dimensional phase portrait of this solution is shown in Fig. 5(a). Here the solution usually takes the form of a modulated dipole with solutions coming very close to the invariant axis $B^2 = 0$. Occasionally (about 10 per cent of the time) the solution flips to a quadrupole mode as it emerges from the minimum. This complicated solution is obviously a product of both Type 1 and Type 2 modulation, but the Type 1 modulation is a secondary effect.

### 3 THE ODE MODEL: FORMULATION

We start the derivation of the new system by considering the equations describing the interaction of oscillatory dipole and quadrupole modes under the action of instantaneous non-linearities (KL96). Here $z_1$ represents the dipole mode whilst $z_2$ is the quadrupole mode. These complex amplitudes correspond to $z_o$ and $z_e$ respectively in the treatment of KL96. The trivial (zero-field) state becomes unstable to these oscillatory modes at a Hopf bifurcation. The evolution of each separate mode is then described...
that resonances will be important and so we set the rates and frequencies of the two modes are similar. This implies two Hopf bifurcations occur close together and that the growth of quadrupole modes, respectively, and \( q \) near onset) by the normal form for a Hopf bifurcation, i.e.

\[
\dot{z}_1 = (\mu + \omega_1) z_1 + \alpha |z_1|^2 z_1, \\
\dot{z}_2 = (\mu + \omega_2) z_2 + \alpha' |z_2|^2 z_2,
\]

where \( \mu, \alpha > 0 \) are the linear growth rates of the dipole and quadrupole modes, respectively, and \( \omega_1, \omega_2 \) are their frequencies. Hence the dipole mode is preferred if \( \alpha > 0 \) with the quadrupole being the more unstable if \( \alpha < 0 \). The non-linear equilibration and evolution of the pure modes is controlled by the complex coefficients \( \alpha \) and \( \alpha' \). The results of the PDEs indicate that the two Hopf bifurcations occur close together and that the growth rates and frequencies of the two modes are similar. This implies that resonances will be important and so we set \( \alpha \) small and \( \omega_1 \approx \omega_2 \). The interactions between the two modes are provided by the addition of further cubic terms that model the resonances of the two modes whilst respecting the symmetries of the system (see the Appendix or KL96 for details). With these additional terms, the equations for the evolution of these modes now take the form

\[
\dot{z}_1 = (\mu + \sigma + i \omega_1) z_1 + \alpha |z_1|^2 z_1 + b |z_2|^2 z_1 + c z_1^* z_1, \\
\dot{z}_2 = (\mu + i \omega_2) z_2 + \alpha' |z_2|^2 z_2 + b' |z_1|^2 z_2 + c' z_1^* z_2,
\]

where an overbar denotes the complex conjugate. The complex coefficients \( b, b', c \) and \( c' \) quantify the non-linear couplings of the two modes. As discussed by Knobloch & Landsberg (KL96) this fourth-order system is consistent with the symmetries of a rotating star. In addition to the pure dipole and quadrupole solutions, the system now possesses mixed-mode solutions (periodic but asymmetric dynamos), as well as several different types of quasiperiodic dynamos, of both pure and mixed parity.

The driving of velocity perturbations by the magnetic field (both dipole and quadrupole modes) can now be included in the model by decomposing the velocity into components that are symmetric and antisymmetric with respect to the equator, and then developing evolution equations for these two components. In the absence of any magnetic field both the symmetric velocity (which we shall denote by \( \varpi \)) and the antisymmetric velocity (\( \omega \)) decay. These velocities are sustained only by the action of the non-linear Lorentz force and so the evolution equations truncated at first order must take the form

\[
\dot{\varpi} = -\tau_1 \varpi, \\
\dot{\omega} = -\tau_2 \omega;
\]

thus the decay rates are controlled by the real parameters \( \tau_1 \) and \( \tau_2 \).

The non-linear Lorentz force is now included by the addition of quadratic terms to equations (5). In principle a great variety of quadratic terms involving combinations of the complex fields \( z_1 \) and \( z_2 \) are available to us, but many of these can be ruled out on the basis of symmetry considerations. A full discussion of the possible terms and the reasons for their inclusion or exclusion is included in the Appendix. For now it suffices to realize (as noted in Section 2) that pure modes (i.e. dipole or quadrupole modes) may only generate symmetric velocity perturbations. However, mixed mode solutions (i.e. solutions with both dipole and quadrupole components) must generate antisymmetric perturbations too. We therefore include quadratic terms in equations (5) to yield

\[
\dot{\varpi} = -\tau_1 \varpi + e_1 (|z_1|^2 + |z_2|^2), \\
\dot{\omega} = -\tau_2 \omega + e_2 (z_1 z_2^* + z_2 z_1^*),
\]

where \( e_1 \) and \( e_2 \) are real coefficients. Notice that the antisymmetric velocity \( \omega \) is driven only if both \( z_1 \) and \( z_2 \) are non-zero.

The velocity perturbations now act so as to generate magnetic fields. A straightforward consideration of the symmetries of the system now possesses mixed-mode solutions (periodic but asymmetric dynamos), as well as several different types of quasiperiodic dynamos, of both pure and mixed parity.

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The velocity perturbations now act so as to generate magnetic fields. A straightforward consideration of the symmetries of the
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system with respect to the equator (see the Appendix) leads immediately to the conclusion that the dipole component of the field can be produced either by the action of the symmetric velocity \( v \) on the dipole field \( z_1 \), or by the antisymmetric velocity \( w \) acting on the quadrupole field \( z_2 \). We include both possibilities by adding terms of the form \( f_1(v, w^2)z_1 + g_1(v, w^2)wz_1 \) to the equation for \( z_1 \) and, similarly, \( f_2(v, w^2)z_2 + g_2(v, w^2)wz_1 \) to the equation for \( z_2 \). Here \( f_1, f_2, g_1 \) and \( g_2 \) are arbitrary (complex) functions of \( v \) and \( w^2 \) which we expand about \( v = w = 0 \) so that

\[
\begin{align*}
    f_1 &= \eta + \epsilon v + \delta v^2 + \kappa w^2, \\
    f_2 &= \eta' + \epsilon' v + \delta' v^2 + \kappa' w^2, \\
    g_1 &= \beta + \gamma v, \\
    g_2 &= \beta' + \gamma' v.
\end{align*}
\]

(7)

The details of the rationale behind the choice of these terms are given in the Appendix.

In the following we absorb the coefficients \( \eta, \eta' \) into \( \mu, \sigma \). The resulting sixth-order system is thus

\[
\begin{align*}
    \dot{z}_1 &= (\mu + \sigma + i \omega_1)z_1 + a |z_1|^2 z_1 + b |z_2|^2 z_1 + c \bar{z}_2 \bar{z}_1, \\
    &\quad + (\epsilon v + \delta v^2 + \kappa w^2)z_1 + (\beta + \gamma v) wz_2, \\
    \dot{z}_2 &= (\mu + i \omega_2)z_2 + a' |z_2|^2 z_2 + b' |z_2|^2 z_2 + c' \bar{z}_2 \bar{z}_2 \\
    &\quad + (\epsilon' v + \delta' v^2 + \kappa' w^2)z_2 + (\beta' + \gamma' v) wz_1, \\
    \dot{v} &= -\tau_1 v + e_1 (|z_1|^2 + |z_2|^2), \\
    \dot{w} &= -\tau_2 w + e_2 (\bar{z}_2 \bar{z}_2 + \bar{z}_1 \bar{z}_1).
\end{align*}
\]

(8)

This system possesses two invariant subspaces, corresponding to pure dipole and quadrupole solutions. It is instructive to consider the behaviour in these invariant subspaces. If the dipole mode is considered in isolation and the quadrupole energy is set to zero, the antisymmetric velocity \( w \) decays to zero (as it needs both field components to be sustained) and the order of the system is reduced. Formally, setting \( z_2 = w = 0 \) in equation (8) simplifies the system dramatically and yields

\[
\begin{align*}
    \dot{z}_1 &= (\mu + \sigma + i \omega_1)z_1 + a |z_1|^2 z_1 + (\epsilon v + \delta v^2)z_1, \\
    \dot{v} &= -\tau_1 v + e_1 |z_1|^2.
\end{align*}
\]

(9)

(Similarly, setting \( z_1 = w = 0 \) confines the solutions to the quadrupole subspace where the corresponding evolution equations for \( z_2 \) apply.) The system is third-order and is a version of the simple ‘one-fixed-point model’ derived by Tobias et al. – which can in turn be related to the normal form equations for a saddle-node/Hopf bifurcation. Equation (9) is the same as equation (17) of TWK95 except that here \( a \) is allowed to be complex. Since (9) preserves axisymmetry (i.e. invariance under \( z_1 \rightarrow z_1 e^{i\theta} \) the solutions are either periodic or quasiperiodic. In TWK95 this symmetry was broken by the addition of a cubic term to allow chaotic modulation.

Here we allow the interaction between the modes to produce the chaotic behaviour. Such behaviour arises in one of two ways. In the first, explored by KL96, the large-scale velocity components \( (v, w) \) play a passive role, and complex dynamics requires \( c_\epsilon \neq 0 \), although quasiperiodic solutions can be found even if this condition is violated. In Fig. 6 we show a quasiperiodic solution obtained with the coefficients used to generate figure 5(a) of KL96; thus we take the primed quantities to be equal to their unprimed counterparts and set the coefficients \( \epsilon, \beta, \gamma, \delta \) and \( \kappa \) to zero, so that the new interactions are switched off. In addition we have set \( \tau_1 = 1.0, \tau_2 = 1.0, e_1 = 0.1, e_2 = 0.001 \), thereby allowing the magnetic field to drive a velocity perturbation (even though this velocity does not affect the field). This procedure allows us to project the solution on to a three-dimensional space spanned by the total dipole energy \( E_D = |z_1|^2 \), the total quadrupole energy \( E_Q = |z_2|^2 \) and

Figure 7. Case 1: time series for dipole solutions, showing \( |\text{Re}(z_1)|^2 \) (solid line) and \( |z_1|^2 \) (dashed line) versus \( t \) for (a) weakly modulated \( (\mu = 0.59) \) and (b) strongly modulated \( (\mu = 0.63) \) quasiperiodic solutions. The toroidal field (including the basic cycle) is represented by \( |\text{Re}(z_1)|^2 \) whilst \( |z_1|^2 \) yields the envelope of activity.

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Table 1. Parameters for the various cases considered here. (The remaining parameters remain unchanged and are defined in the text.)

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<th>$\beta'$</th>
<th>$\delta$</th>
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Figure 8. Case 1: time series for dipole solutions with deep minima ($\mu = 0.652$). (a) $|\text{Re}(z_1)|^2$ (solid) and $|z_1|^2$ (dashed), (b) $|\text{Re}(z_2)|^2$ (solid) and $|z_2|^2$ (dashed), (c) $w$ and (d) $v$. The dipole energy is strongly modulated by Type 2 modulation driven by the symmetric velocity $v$. The quadrupole energy tends to zero and so, consequently, does the antisymmetric velocity component $w$. 

the symmetric part of the velocity $v$, and hence to compare the resulting solution with Fig. 3(a) obtained from the PDEs. In this projection periodic solutions of the ODEs appear as fixed points while quasiperiodic solutions appear as limit cycles, i.e. the limit cycle in Fig. 6 corresponds to a two-torus in the PDEs. This solution is evidently of Type 1: in both cases energy is exchanged between the dipole and quadrupole modes with little change in the velocity perturbations $v$. For other values of the coefficients modulation in the form of periodic or chaotic ‘bursts’ is possible (KL96, Moehlis & Knobloch 1998), but the parity of the solutions remains near zero throughout. Such solutions, called Type II by KL96, do not appear to be relevant to the present Sun (although they may be to other stars) and are eliminated in what follows by setting $c = c' = 0$.

In the next section we focus instead on the new interactions allowed by the inclusion of the Malkus–Proctor effect in the sixth-order model.

4 RESULTS

In this section we demonstrate that the behaviour found in the PDE model of Section 2 is reproduced in solutions of the sixth-order system (8) introduced above. These ODEs contain eight real and sixteen complex coefficients so it would be futile to attempt to explore the parameter space systematically. Instead we observe that in mean-field dynamo models the Hopf bifurcations for the dipole and quadrupole modes typically occur close together and with very similar frequencies. We therefore set $\omega_1 = \omega_2$ and choose a small value of $\sigma$. For simplicity, we set $c$, $c'$, $\kappa$, $\kappa'$, $\gamma$ and $\gamma'$ equal to zero and then choose $\omega_1 = 12.51081$, $\omega_2 = 12.51081$, $a = 0.5 - 0.5i$ and $\tau_1 = 1.0$. We regard $\mu$ as a distinguished parameter (the control parameter) and consider the five sets of values for the remaining coefficients that are displayed in Table 1. These cases have been chosen because they exhibit states that can be directly related to the PDE results already described.

For Case 1, $\sigma > 0$ and so the initial Hopf bifurcation gives rise to an oscillatory dipole solution. For all values of $\mu$ that we have studied, there exist pure dipole solutions that are stable, i.e. the dipole subspace is attracting. Within this subspace we can reproduce the sequence of transitions described by TWK95. As $\mu$ is increased the non-linear periodic solution undergoes a secondary Hopf bifurcation giving rise to a weakly modulated quasiperiodic oscillation as shown in Fig. 7(a). A further increase in $\mu$ yields strongly modulated solutions as in Fig. 7(b). For yet larger values of $\mu$ we find nearly heteroclinic, modulated behaviour. In the time series in Fig. 8 the initial conditions have both dipole and quadrupole components but the transient quadrupolar oscillations – and hence the antisymmetric velocity component $w$ – decay to zero. In this case we can see that the modulation is driven by the symmetric velocity $v$, which varies on the time-scale of the modulation. Fig. 9(a) projects the trajectory on to the three-dimensional phase space $(E_D, E_Q, v)$ that was introduced for Fig. 6. The trajectory is attracted towards the dipole subspace and spirals out to form a strongly modulated orbit. Thus this model displays the same bifurcation sequence as was found for the PDEs when dipole symmetry was imposed (Tobias 1996). The absence of chaotic dipoles can be attributed to the simplification of the system within the dipole subspace (as discussed in the derivation of the system in Section 3). The addition of higher-order terms that break the symmetry was imposed (Tobias 1996). The absence of chaotic attractors (TWK95).

The phase portrait in Fig. 9(b) shows a trajectory with the same initial conditions but for Case 2. Here the sign of $\sigma$ is reversed (with all the other parameters as for Fig. 8(a)) and the oscillatory quadrupole mode appears first. Now the trajectory spirals towards the $(E_Q, v)$ plane and is attracted to a modulated quadrupole solution. Again, as the solution is of pure parity, the antisymmetric velocity $w$ decays to zero and the modulation is driven by $v$.

Since we have so far chosen the coefficients $b, b', c$ and $c'$ to be zero, the only coupling between $z_1$ and $z_2$ is through $v$ and $w$. Because $w$ represents the antisymmetric part of the velocity it acts so as to generate $z_1$ from $z_2$ and vice versa. Indeed all mixed modes appear to be unstable if $w$ is omitted (for $b = b' = c = c' = 0$). Increasing $e_2$ leads to larger values of $w$ and so increases the coupling between $z_1$ and $z_2$. This favours the existence of strongly modulated mixed mode solutions as shown in Fig. 10 where the time series for Case 3 (with $\mu = 1.03$) are displayed. Here the dipole and quadrupole energies are of the same order and both are modulated by the Malkus–Proctor effect. Furthermore Fig. 10(c,d) shows that the antisymmetric velocity component $w$ is of the same order as $v$. In order to compare results with those for the PDEs in Fig. 2, we introduce the total energy $E = |Re(z_1)|^2 + |Re(z_2)|^2$ and its envelope $E_{KL} = |z_1|^2 + |z_2|^2$, together with the parity $P_{KL} = (|z_2|^2 - |z_1|^2)E_{KL}$ (cf. KL96). Fig. 11(a,b) shows the evolution of these quantities for Case 3 whilst the corresponding phase portrait follows in Fig. 11(c). The trajectory spirals in to a
mixed mode limit cycle. The energy is strongly modulated while the parity $P_{KL} < 0$. During the deep minima both $v$ and $w$ decay exponentially at rates determined by $\tau_1$ and $\tau_2$ and the trajectory approaches the neighbourhood of the origin. This is a clear example of Type 2 modulation for a mixed mode solution, in contrast with the Type 1 modulation shown in Fig. 2 for the PDEs.

For Case 4 we move away from the degenerate system in order to illustrate more general patterns of behaviour: specifically, $a \neq a'$, $b \neq b'$ and $\beta \neq \beta'$. Moreover, the non-linear coefficients are chosen to be very different from those in Cases 1–3 so that a wide range of behaviour can be demonstrated. The two sets of results in Fig. 12 are for the same value of $\mu = -0.205\,009$ but have different initial conditions. In Fig. 12(a) the time series settle down to a quasiperiodic dipole with $P_{KL} = -1$ and $w = 0$. The corresponding phase portrait shows the approach to a periodic orbit in the dipole subspace. However, with initial conditions such that $P_{KL} = 0$, the transient modulation dies away to leave a periodic mixed mode solution (Fig. 12b). The trajectory in phase space at first appears to approach an unstable mixed mode two-torus, which appears as a limit cycle in this projection, and then spirals in to the periodic orbit, which appears as a fixed point. These results demonstrate that the system possesses a plurality of solutions with the form of fixed

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**Figure 10.** Case 3: time series for Type 2 modulation of a mixed-mode solution. The time-series are as for Fig. 8. Here both the dipole and quadrupole energies undergo Type 2 modulation. The solution therefore has mixed parity and hence the antisymmetric part of the velocity ($w$) no longer decays to zero.
points, limit cycles, tori etc., which may be either attracting or repelling. Moreover, there are at least two very different stable solutions. This seems to be a common feature of non-linear dynamos whether described by PDEs or by low-order models. Indeed, some truncated mean field dynamo models (Covas et al. 1997; Covas & Tavakol 1997) exhibit much more extreme sensitivity to initial conditions.

Finally, we consider Case 5 with a strongly asymmetric choice of coefficients. In this case there is a complicated sequence of transitions. As $\mu$ is increased the solutions change from periodic dipoles to quasiperiodic dipoles until the modulation becomes chaotic and the solution exhibits minima. As shown in Fig. 13, a further increase in $\mu$ yields solutions in which the field is dipolar when strong but mixed when it is weak. These solutions are analogous to those found in the PDE model by Tobias (1997a).

For $\mu = 1.765$ we obtain a solution that shows strong Type 2 modulation of both dipole and quadrupole energies (Fig. 14a,b). The parameters are such that $E_D$ is usually larger than $E_Q$ (although both energies spend much time in minima) so that the parity $P_{KL} \approx -1$; more precisely, $E_Q \ll E_D$ when $E_D$ is large but the parity rises during grand minima (when both energies are small). The important feature, however, is that the parity occasionally flips after a prolonged and deep grand minimum. For instance the solution enters the long minimum starting at $t = 490$ with a predominately dipole field but emerges as a strong quadrupole. This is exactly what was found for the PDE solution shown in Figs 1(b) and 5(a). In the corresponding phase portrait for the ODEs (Fig. 5b) the basic cycle is filtered out, leaving a much tidier pattern. Apart from that, the similarity between the two trajectories in Fig. 5 is truly remarkable.

The chaotic trajectory in Fig. 5(b) frequently approaches the $v$ axis and moves down it as the velocity decays, until the trajectory is extremely close to the origin. This implies the presence of a nearby homoclinic bifurcation. This bifurcation is responsible not only for
chaotic behaviour, as argued by TWK95, but also for allowing transitions between the dipole and quadrupole subspaces. When $E_D$, $E_Q$, $v$ and $w$ are all small, behaviour in the immediate vicinity of the origin determines the parity with which a trajectory emerges. It follows that the addition of a small amount of noise could have a profound qualitative effect on the nature of the solutions. Stochastic effects have been studied in other systems with slow–fast dynamics (e.g. Stone & Holmes 1989; Hughes & Proctor 1990; Lythe & Proctor 1993; Lythe & Tobias, in preparation). When trajectories approach an invariant subspace (such as the $v$-axis here) and spend a long time in its vicinity, the effect of noise depends not on its amplitude $\Delta$ but on $\mu | \ln \Delta |$. Hence weak stochastic disturbances can exert a major influence on trajectories when they eventually escape. In this case, therefore, we would expect noise to affect the parity, the duration of grand minima and the amplitude of modulation.

Figure 12. Sensitive dependence on initial conditions: $E(t)$, $P(t)$ and the phase portrait for Case 4, showing two different attractors: (a) a quasiperiodic dipole, and (b) a periodic mixed mode solution.
through the effects of pumping and magnetic buoyancy. It, as well as to an enhancement of the effective magnetic diffusivity tachocline and the way that angular momentum is transferred within the value of the turbulent viscosity may be related to the structure of the magnetic field of either parity and so flipping can occur. Moreover, such perturbation. After a grand minimum, the energy can be given back when energy is transferred from the magnetic field to the velocity field of the Sun is predominantly dipolar. At the end of the Maunder Minimum, however, the field was markedly asymmetric and spots were almost all confined to one hemisphere during the low-amplitude cycle that reached its maximum in 1706 (Ribes & Nesme-Ribes 1993). Such behaviour is consistent with our results and suggests, moreover, that parity flipping may have occurred in the past (Beer et al. 1998). If so, it should be associated with deep and prolonged grand minima – so Spörer rather than Maunder minima (Stuiver & Braziunas 1988) are the most likely candidates.

The patterns of behaviour in both ODE and PDE models change as the stability parameter ($\mu$ or the dynamo number $|D|$) increases. For a star we expect that $|D|$ is a monotonically increasing function of the angular velocity $\Omega$, e.g. $|D| \propto \Omega^2$ (Pallavicini 1996). As the star evolves it loses angular momentum through magnetic braking and therefore gradually spins down. Hence $\Omega$ and $|D|$ decrease with

5 DISCUSSION

We have succeeded in showing that the two types of modulation already found for the PDEs describing a particular mean field dynamo are also present in our low-order system of ODEs. The advantage of introducing such an idealized model is that the results obtained are generic and therefore likely to be robust. Hence we may expect to find similar behaviour not only in a wide class of mean field dynamos but also in self-consistent non-linear dynamos – and indeed in real stars. For example, the model we present here is capable of describing behaviour found not only in mean field models using the Malkus–Proctor effect, but also in those that rely on quenching, whether of $\alpha$, $\omega$ or $\eta$, or on buoyancy non-linearities. Moreover, the phenomena of on-off intermittency and crisis intermittency, as described by Covas & Tavakol (1997), can be found in this model too.

One of our principal aims has been to clarify the distinction between Type 1 modulation, when energy is simply exchanged between the dipole and quadrupole modes, and Type 2 modulation, when energy is transferred from the magnetic field to the velocity perturbation. After a grand minimum, the energy can be given back to a field of either parity and so flipping can occur. Moreover, such grand minima can only occur if there is a delay in the quenching mechanism (cf. Yoshimura 1978); in our models, the time-scale on which energy is returned to the field is controlled by $\tau$, and prolonged grand minima arise when $\tau \ll 1$. The relatively low value of the turbulent viscosity may be related to the structure of the tachocline and the way that angular momentum is transferred within it, as well as to an enhancement of the effective magnetic diffusivity through the effects of pumping and magnetic buoyancy.

It should be stressed that no models with simple $\alpha$ quenching have been found that display the phenomena we describe here as grand minima (i.e. Type 2 modulation as exhibited by the Sun). With quenching mechanisms the energy can only be transferred between modes of different symmetries and that leads to changes in the parity of solutions. Therefore models with minima in energy that solely include $\alpha$ quenching must be exhibiting Type 1 modulation, i.e. modulation arising from changes in parity.

Throughout this paper we have presumed that the modulation is deterministic, and that it corresponds to quasiperiodic behaviour following a secondary Hopf bifurcation. Nevertheless, we should emphasize that behaviour during deep grand minima is strongly affected by stochastic disturbances. In particular, noise facilitates transitions between the dipole and quadrupole subspaces. So the flipping illustrated in Fig. 4 should be encouraged by the stochastic effects of turbulent convection.

Our results have clear implications for cyclic activity in stars (Weiss & Tobias 1997). We should expect to find both parity changes and modulation, of Types 1 and 2. Modulation associated with changes in parity has not so far been observed in the Sun, and would be hard to detect in other stars. On the other hand, the Maunder Minimum was a clear example of Type 2 modulation of solar activity, while the proxy record shows that grand minima have recurred over the last 10 000 yr. Furthermore, there is evidence of similar behaviour in other late-type stars. Systematic observations of sunspots over the last four centuries indicate that the magnetic field of the Sun is predominantly dipolar. At the end of the Maunder Minimum, however, the field was markedly asymmetric and spots were almost all confined to one hemisphere during the low-amplitude cycle that reached its maximum in 1706 (Ribes & Nesme-Ribes 1993). Such behaviour is consistent with our results and suggests, moreover, that parity flipping may have occurred in the past (Beer et al. 1998). If so, it should be associated with deep and prolonged grand minima – so Spörer rather than Maunder minima (Stuiver & Braziunas 1988) are the most likely candidates.

The patterns of behaviour in both ODE and PDE models change as the stability parameter ($\mu$ or the dynamo number $|D|$) increases. For a star we expect that $|D|$ is a monotonically increasing function of the angular velocity $\Omega$, e.g. $|D| \propto \Omega^2$ (Pallavicini 1996). As the star evolves it loses angular momentum through magnetic braking and therefore gradually spins down. Hence $\Omega$ and $|D|$ decrease with
The earlier magnetic history of the Sun can be investigated by studying younger stars that rotate more rapidly and are more active, subject to the caveat that the structure of convection and differential rotation may be quite different when $Q$ is large (Weiss 1996). We predict that flipping should become more frequent as $Q$ increases, and that the field will spend less time with any given parity. Looking to the long-term future of the Sun, we expect that flipping will become much rarer as $Q$ decreases; there will then be a decrease in the extent of modulation until there is a cyclic mode with a pure parity; thereafter, any symmetry breaking will lead to Type 1 modulation.

The present-day Sun has a magnetic field with dipole symmetry, so it is natural to suppose that other similar stars have dipole fields too. Our investigations show, however, that fields with quadrupole or mixed symmetry are also likely to be found. As the Sun evolved and spun down – and as it continues to evolve in the future – it is likely to have passed through phases when its field was predominantly quadrupolar or had mixed parity. Whether the toroidal field is symmetric or antisymmetric has little effect on butterfly diagrams for sunspots. Nevertheless, since parity changes are only likely to occur at deep grand minima, we may presume that the solar magnetic field has retained its present symmetry since the end of

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**Figure 14.** Time-series for flipping between different parities. (a) $|\text{Re}(z_1)|^2$ (solid) and $|z_1|^2$ (dashed), (b) $|\text{Re}(z_2)|^2$ (solid) and $|z_2|^2$ (dashed), (c) $\Sigma(t)$ and (d) $\mathcal{P}(t)$, all for Case 5 with $\mu = 1.765$. Both the dipole and quadrupole energies are chaotically modulated, with the dipole energy generally larger than the quadrupole. Occasionally, after a grand minimum, the parity of solutions can flip to a quadrupole mode in a manner reminiscent of the behaviour found in the PDEs. The corresponding phase portrait is in Fig. 5(b).
the Maunder Minimum. Given the observed asymmetry of its field then, there is, however, no justification for supposing that the Sun’s field has had dipole symmetry throughout its recent history, or that it will retain that symmetry in the future.

ACKNOWLEDGMENTS

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APPENDIX: DERIVATION OF THE LOW-ORDER MODEL

We explain here how the sixth-order system (8) can be derived for a stellar dynamo, referred to spherical polar co-ordinates \((r, \theta, \phi)\). Exactly the same treatment applies to a Cartesian model, with \(x = (2L/\pi)\theta\) and \(y\) ignorable. Near the onset of the dynamo instability, the axisymmetric toroidal magnetic field is well approximated by the expression

\[
B_t(r, \theta, \phi) = \Re\{z_1f_1(r, \theta) + z_2f_2(r, \theta)\}
\]

+ higher order terms,

where \(f_{1,2}\) denote the spatial eigenfunctions of the dipole and quadrupole modes, respectively. The eigenfunctions \(f_{1,2}\) have the following properties under reflection in the equator: \(f_1(r, \pi - \theta) = -f_1(r, \theta)\), \(f_2(r, \pi - \theta) = f_2(r, \theta)\). There is a similar expression for the poloidal field with amplitudes that are related to \((z_1, z_2)\) by the solution of the linear dynamo problem. Consequently the dynamics of both fields is fully determined once the time evolution of \((z_1, z_2)\) is known. From the reflection invariability in the equator it follows that the equations obeyed by the complex amplitudes \(z_1, z_2\) must be unchanged under the operation reflection: \((z_1, z_2) \rightarrow (z_1, -z_2)\).

Motivated by the observation that in mean field dynamo models both dipole and quadrupole modes set in at nearby dynamo numbers and with similar frequencies, we treat the mode interaction problem as a perturbation of a degenerate problem in which they set in simultaneously and with identical frequencies. Such a problem is described as a 1:1 resonance. The non-linear equations for such a resonant double Hopf bifurcation can be simplified by non-linear, near-identity coordinate changes. The resulting simplified equations are called normal form equations, and have an additional symmetry called a phase-shift symmetry. This symmetry takes the form

\[
\text{phase shift}: (z_1, z_2) \rightarrow (z_1 e^{i\varphi}, z_2 e^{i\varphi}), \quad 0 \leq \varphi < 2\pi,
\]

and for nonaxisymmetric modes can be identified with spatial rotations. Equations constructed to be invariant under these operations will reflect the symmetries of a rotating star: if we have one solution, then the solution obtained by reflection in the equator will also be a solution, and similarly for the phase shifts/rotations. To construct such equations we note first that there are four fundamental invariants of these two operations:

\[
\sigma_1 = |z_1|^2, \quad \sigma_2 = |z_2|^2, \quad \sigma_3 = z_1^* z_2^2, \quad \sigma_4 = z_2^* z_1^2.
\]

(4A)

Any invariant polynomial can be written as a function of these four fundamental invariants, which satisfy the relation (a syzygy) \(\sigma_1 \sigma_2 = \sigma_3^2 \sigma_4^2\). To construct a differential equation we also need to know all quantities that transform like \((z_1, z_2)\). All such vectors can be constructed from the fundamental vectors \((z_1, z_2), (z_1^2 z_2, z_1 z_2^2)\). It follows that

\[
\dot{z}_1 = F_1(\sigma_1, \sigma_2, \sigma_3, \sigma_4) z_1 + G_1(\sigma_1, \sigma_2, \sigma_3, \sigma_4) z_2^2
\]

(5A)

\[
\dot{z}_2 = F_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4) z_2 + G_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4) z_1^2,
\]

where \(F_j, G_j, j = 1, 2,\) are (complex) invariant functions.
In the presence of the Malkus–Proctor effect there are two additional variables, the (real) amplitudes $v, w$ of the even and odd parity components of the large-scale velocity. These amplitudes are real because the flow is driven by the quadratic Lorentz force, averaged over the period of the basic cycle. The equations for these velocity components must also respect the symmetries of the rotating star, and hence be left invariant under reflections and phase shifts. These amplitudes contribute additional fundamental invariants $(v, w^2, z_2 z_{1w}, z_1 z_2 w)$, and an additional fundamental vector that transforms as $(z_1, z_2)$, i.e. $(z_2 w, z_1 w)$. It follows that

\[ \dot{z}_1 = F_1(\sigma) z_1 + G_1(\sigma) z_2^2 z_1 + H_1(\sigma) z_2 w, \]

\[ \dot{z}_2 = F_2(\sigma) z_2 + G_2(\sigma) z_2^2 z_2 + H_2(\sigma) z_1 w, \]

where $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, v, w^2, z_2 z_{1w}, z_1 z_2 w)$. The invariant functions $F_1, ..., H_2$ are all complex.

Likewise, there are three (real) vectors that transform like $(\dot{v}, \dot{w})$, i.e. $(1, w)$, $(1, z_2 z_1)$ and $(1, z_1 z_2)$. It follows that the equations for $(v, w)$ must take the form

\[ \dot{v} = F_3(\sigma), \]

\[ \dot{w} = F_4(\sigma) w + \text{Re}\{G_3(\sigma) z_2 z_1 + H_3(\sigma) z_2 z_2\}, \]

with $F_3, F_4$ real and $G_3, H_3$ complex.

If we now expand all the invariant functions in Taylor series and retain only the leading order terms we obtain the equations

\[ \dot{z}_1 = i \omega z_1 + a |z_1|^2 z_1 + b |z_2|^2 z_1 + c z_2^2 z_1 \]

\[ + (ev + b\bar{v} + kw^2) z_1 + (\beta + \gamma v) w z_2, \]

\[ \dot{z}_2 = i \omega z_2 + a' |z_2|^2 z_2 + b' |z_1|^2 z_2 + c' z_1^2 z_2 \]

\[ + (e' v + b' \bar{v} + k' w^2) z_2 + (\beta' + \gamma' v) w z_1, \]

\[ \dot{v} = -\tau_1 v + d_1 v^2 + d_{1w} w^2 + e_1 |z_1|^2 + f_1 |z_2|^2, \]

\[ \dot{w} = -\tau_2 w + d_2 v w + e_2 (z_1 z_2 + z_2 z_1) \]

\[ + f_2 (z_1 z_2 - z_2 z_1). \]

In these equations we have set $F_1(0) = F_2(0) = i \omega, F_3(0) = 0$ to recover the assumed linear problem. In writing equations (6) we have set $d_{1w} = d_{1w} = d_2 = f_2 = 0$ and chose, on physical grounds, $e_2 f_1 > 0$, scaling $z_2$ in such a way that $v$ is driven in response to the total energy $|z_1|^2 + |z_2|^2$. The omitted quadratic terms appear to be inessential for the type of dynamics discussed here. Four additional (real) coefficients may be set to $\pm 1$ by appropriate scaling, but we choose not to do so. Finally, it remains to unfold these equations by restoring the slightly different growth rates and frequencies of the dipole and quadrupole modes, as in equations (8).

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