Baroclinic instability in stars

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Summary. The conditions for the occurrence of adiabatic baroclinic instability in differentially rotating stars are studied using the quasi-geostrophic approximation on a $\beta$-plane. In rapidly rotating stars (near breakup) the instability can occur throughout the star. In slowly rotating stars, it can only occur in very narrow regions located near (a) the center of the star, (b) the boundary of a convection zone, (c) a jump in the molecular weight or in the buoyancy frequency, or (d) a jump in the gradient of the rotation rate. We conclude that the instability cannot be responsible for the mixing needed to explain the Li abundance anomalies in main sequence stars or the low neutrino flux of the Sun. It could, however, be important for mixing near the cores of red giants.

Key words: differential rotation – hydrodynamic stability – mixing – red giants

1. Introduction

In a differentially rotating star, which we assume to be convectively stable, a number of hydrodynamic instabilities can occur (Zahn, 1975; Tassoul, 1978; Knobloch and Spruit, 1982). The interest in such instabilities is stimulated, on the one hand, by their potential for transporting angular momentum and hence influencing the distribution of the rotation speed throughout the star. By measuring the rotational splitting of global nonradial oscillations of the star, this distribution can in principle be determined observationally. The presence of such a splitting has been implied for $\beta$-cephei stars (Osaki, 1971, 1974); white dwarfs of ZZ Ceti type (Robinson, 1979), Ap stars (Kurtz, 1983) and the sun (Claverie et al., 1981; Gough, 1982). On the other hand, the instabilities can also cause slow mixing in the radiative cores of stars. Such mixing could have significant effects on the evolution of stars and their surface abundances (Genova and Schatzman, 1979; Schatzman and Maeder, 1981; Maeder, 1982; Lambert and Ries, 1981; Bienaymé et al., 1983; see also the review by Scalo, 1981).

In rapidly rotating stars a larger number of instabilities can occur than in slowly rotating stars. A critical parameter in this connection is the ratio of the rotation rate $\Omega$ to the Brunt-Väisälä frequency $N$. If this ratio is small, only instabilities which manage to circumvent the strongly stabilizing effect of the thermal stratification (measured by $N$) can occur. One such instability is the adiabatic baroclinic instability, in which the motions are mostly horizontal. This is the instability which on earth governs the behavior of the Gulf stream and the formation of weather fronts (e.g. Pedlosky, 1979; Hoskins, 1982). Shear instability can also occur if turbulence can be maintained on such small scales that radiative exchange between turbulent elements eliminates thermal buoyancy forces on these elements (Townsend, 1958; Zahn, 1975, 1983). Finally, diffusive (secular) instabilities can occur, whose presence depends in an essential way on radiative exchange (Goldreich and Schubert, 1967; Fricke, 1968; McIntyre, 1970a; Acheson, 1978; Knobloch and Spruit, 1982, 1983).

The presence and efficiency of these instabilities is affected in an important way by a stratification in the molecular weight. Even a relatively weak inward increase of molecular weight is sufficient to suppress the shear and Goldreich-Schubert-Fricke (GSF) instabilities. The mixing effects needed to explain the low solar neutrino flux and the abundance anomalies in low-mass red giants are required to work in the presence of such molecular weight gradients. This leaves only the baroclinic instability and a diffusive instability, called the ABCD instability (Knobloch and Spruit, 1983), as candidates.

It has been stated by Tassoul (1983) that baroclinic instability is "ever present". If this were true it would make the instability of considerable interest in stellar evolution. In a previous paper (Knobloch and Spruit, 1982) we argued, on other basis of a model of rather restricted applicability that the adiabatic baroclinic instability would not occur in slowly rotating stars. The aim of the present paper is to give as complete a discussion of this issue as is possible. We base our discussion on a sufficient condition for stability that is well known to geophysicists. Our conclusion is that in a slowly rotating star the instability can still occur but only in very restricted regions of the star. These conclusions are similar to those of Zahn (1983).

To avoid confusion in what follows we reserve the name "baroclinic instability" for the adiabatic, nonaxisymmetric instability studied in geophysics. The GSF and ABCD instabilities also depend on the baroclinicity of the basic state, but their behavior is so different from that of the adiabatic instability that it is preferable to have separate names for them.

2. Mechanism of the baroclinic instability

Good discussions of the mechanism of baroclinic instability have been given, for example by McIntyre (1970b) and Pedlosky (1979). We summarize some essentials here, in a language adapted to astrophysical usage.
Let $(\sigma, \phi, \zeta)$ be a cylindrical coordinate frame aligned with the rotation axis, and $\Omega(\sigma, \zeta)$ the rotation rate. In general, surfaces of constant entropy and constant pressure do not coincide, i.e. the star is baroclinic. In steady state this situation is described by the thermal wind relation. From the equations of hydrostatic balance in the $\sigma$ and $\zeta$ directions one can easily show that (e.g. Knobloch and Spruit, 1982):
\[ 2\Omega \sigma \delta \Omega = N^2 \sin(A - \theta), \]
where $N$ is the Brunt-Väisälä frequency, $A$ the direction of the effective gravity (approximately equal to the latitude) and $\theta$ the angle between the entropy gradient and the equatorial plane. The geometry is sketched in Fig. 1. A fluid element displaced along a surface of constant entropy develops no temperature difference with respect to its surroundings and hence experiences no net buoyancy force. In the absence of additional forces, a slightly perturbed fluid element moves along this surface in circles, under the influence of the Coriolis force, and oscillates about it at the Brunt-Väisälä frequency. It does not gain or lose energy. However, if there exists an additional constraint on the motion which forces it to be more nearly horizontal, energy can be released. From Fig. 1 one can easily see that fluid elements moving in a direction within the wedge $\delta \equiv A - \theta$ experience a buoyancy force which has a component in the direction of motion, and hence are accelerated. The constraint which produces an instability in this way can be, for example, the presence of a solid horizontal boundary. This is the most important constraint in the earth's atmosphere. In stars, a jump in molecular weight can play the same role. Other possibilities are a jump in the Brunt-Väisälä frequency, or a jump in the gradient $\sigma \Omega$ of the rotation rate. At sufficiently high rotation rates $\Omega \sim N$, even gradual variations in $\Omega$ can set off the instability (cf. the discussion in Sect. 4.1).

Since motions within the wedge $\delta$ gain energy from the entropy gradient in this direction, the instability is often referred to as a form of “horizontal convection”. The result of the instability is a reduction of the baroclinicity. Since this is possible only by a change in $\sigma \Omega$, the instability can equally be regarded as a form of shear instability, feeding on the velocity gradient along the $\zeta$-axis.

### 3. Geostrophic approximation

We consider here the stability of differential rotation to motions with a horizontal length scale $L$ and a vertical length scale $D$, such that $D \ll L \ll r$, where $r$ is the radial coordinate. The natural scales of the instability in a slowly rotating star satisfy this relation (cf. Sect. 4.2). For such motions, the so-called geostrophic approximation on a $\beta$-plane can be used, as we will show presently. Of course, if stability is found for such scales, stability against motions on different scales still has to be considered (cf. Sect. 4.2.).

Across a volume of horizontal extent $L$ and vertical extent $D$ the typical variation, $U$, of the fluid velocity due to differential rotation is
\[ U = \max \left( \frac{D}{L} \frac{\partial \Omega}{\partial A} \left| \frac{D}{L} \frac{\partial \ln \Omega}{\partial A} \right| \right). \]
(2)

where $A$ is the latitude and $\Omega(r, A)$ the rotation rate. We define the Coriolis parameter $f$ by
\[ f = 2\Omega \sin A; \]
(3)

then the Rossby number is
\[ Ro = \frac{U}{f L} = \frac{1}{2 \sin A} \max \left( \frac{\partial \ln \Omega}{\partial A} \frac{D}{L} \frac{\partial \ln \Omega}{\partial \ln r} \right). \]
(4)

The latitude variation of $\Omega$ is important for barotropic instabilities (e.g. Watson, 1980) but is not essential for the baroclinic instability, although the presence of horizontal shear modifies the instability, reducing its growth rate and meridional wavelength (Pedlosky, 1964; Stone, 1969). Thus, without loss of generality, we may assume $\partial \ln \Omega / \partial A$ to be small. Concerning the radial variation we assume that $\Omega$ varies smoothly, so that $\partial \ln \Omega / \partial \ln r$ is of order unity. Steeper gradients may occur locally but are less interesting since we are primarily interested in instabilities that occur throughout the star. Thus we find
\[ U = D \Omega; \quad Ro = \frac{D}{2L \sin A}. \]
(5)

It follows that $Ro$ is small except near the equator, where $\sin A \rightarrow 0$. The fact that $Ro$ and $D / L$ are small allows us to use the geostrophic and hydrostatic approximations, respectively. In the following we give a brief summary of the equations in this approximation, following Pedlosky (1979, Chap. 6). We define local cartesian coordinates $(x, z)$, related to spherical coordinates $(\phi, A, r)$, where $\phi$ is the longitude (positive in eastward direction) and $A$ the latitude (positive in the northward direction; note that $A$ corresponds to $\theta$ in Pedlosky), by
\[ x = (\phi - \phi_0) \cos A \xi_0, \]
\[ y = (A - A_0) \xi_0, \]
\[ z = r - r_0. \]
(6)

Let $\xi_0(z)$ be the density, and $u_0 = (u_0(y, z), 0, 0)$ be the flow speed in the unperturbed state. Let $\phi$ be the pressure perturbation and $u = (u, v, w)$ the velocity perturbation. Then $u$ satisfies
\[ \xi_0 u = -\partial \phi, \]
(7a)
\[ \xi_0 \phi = \partial \phi \]
(7b)
(geostrophic approximation). This means that the horizontal equations of motion reduce to a balance between the pressure gradient and the Coriolis force. The vertical equation of motion reduces to
\[ \partial \phi = -q \xi_0, \]
(8)
(hydrostatic approximation), where $q$ is the density perturbation. The evolution of the perturbations is governed by the quasi-
The geostrophic approximation breaks down close to the equator. Assuming that the rotation law is symmetric about the equator, the baroclinicity vanishes there and hence there is no available potential energy for a baroclinic instability. This suggests that our conclusions are uniformly valid over the star even though the approximations used are not.

4. Application to stars

4.0. General considerations

The velocity shear \( \partial_z u_0 \) in Eq. (11) plays the same role in the \( \beta \)-plane approximation as the axial rotation gradient \( \partial_z \Omega \). In particular, the condition for hydrostatic balance yields the thermal wind relation

\[
\partial_z u_0 = -\frac{\theta}{f} \partial_z \ln \tilde{T},
\]

where \( T \) is the potential temperature, defined by

\[
\ln \tilde{T} = \frac{1}{T} \ln p - \ln q.
\]

The angle \( \delta \) between horizontal and constant entropy surfaces is

\[
\delta = \frac{\partial \Omega}{\partial \Omega}.
\]

so that (13) can be written as

\[
\rho_0 \partial_z u_0 = \rho_0 \partial_z \ln \tilde{T} = N^2 \delta,
\]

which is the equivalent of Eq. (1). In terms of the baroclinicity \( \delta \), the potential vorticity gradient can be written as:

\[
\partial_z \Pi_0 = \beta + \partial_z u_0 - f \partial_q (q_0 \delta),
\]

For simplicity, we take \( \partial_z u_0 = 0 \) (cf. the discussion in Sect. 3), and write

\[
-\frac{d \ln \Omega}{d \ln r} = \alpha.
\]

By identifying \( \delta \) with the value appearing in Eq. (1) and taking \( \delta < 1 \), we obtain

\[
\partial_z \Pi_0 = \frac{2 \Omega}{\rho_0} \cos \Lambda_0 \left[ 1 + 2 \sin^2 \Lambda_0 \partial_q \left( q_0 \frac{\Omega^2}{N^2} \right) \right],
\]

as the appropriate form of (16) in astrophysical variables. We note that it is strictly valid only when the horizontal and vertical (radial) scales of the instability is sufficiently small. In the following, we drop the subscripts on \( \Omega \), \( r_0 \), and \( q_0 \) and let \( \bar{\Omega}(r) \) denote the density on a spherical surface of radius \( r \).

A theorem due to Charney and Stern (1962) states that it is necessary for instability that \( \partial_z \Pi_0 \) changes sign somewhere. This is evidently possible only if, somewhere in the star,

\[
\frac{\partial \bar{\Omega}}{\partial \bar{\rho}} \left( \frac{\bar{\Omega}^2}{\bar{N}^2} \right) < -\frac{1}{2}.
\]

Let \( H_\rho, H_N, H_\varphi \) be the length scales (in \( r \)) on which \( \bar{\rho}, N \), and \( \varphi \) vary, respectively. Then instability is possible, roughly speaking, in two classes of circumstances:

1. If \( \Omega/N \sim O(1) \), instability is possible for \( H_\rho, H_\varphi, H_N, H_\varphi \sim r \), i.e. for a smoothly varying rotation profile and stratification. If \( \chi \) is
the ratio of the centrifugal to the gravitational force, we have, in
the absence of a molecular weight gradient,
\[ \frac{\Omega^2}{N^2} = \frac{\chi H_q}{r(V_+ - V)}. \]  
\hspace{1cm} (20)

The condition $\Omega/N \sim 1$ is obviously met if $\chi \sim 1$, i.e. when the star
rotates near breakup. If on the other hand, the rotation is slow, 
($\chi \ll 1$) it is still met near a convection zone (where $V \approx V_0$),
and close to the center of the star (where $H_q \to \infty$).

\begin{itemize}
  \item[i)] If $\Omega/N \ll 1$, instability is possible only near locations in the star
where one of the scales $H_q$, $H_q$, $H_q$ is of the order $r \Omega^2/N^2$.
Since, averaged over the entire star, these scales are of the order $r$,
the volume occupied by such locations is small.
\end{itemize}

\subsection{4.1. Rapid rotation}

In the case of rapid rotation condition (19) has to be applied with
caution since the most unstable length scale $L$ of the instability
becomes comparable with $r$. This is so not only for the instability
of a smooth rotation law but also for instabilities occurring near a
jump in the molecular weight gradient, a discontinuity in the rotation
rate or near boundaries (cf. Sect. 4.2). In this case the
$\beta$-plane approximation breaks down. At the same time the
Rossby number becomes of order unity, and $L \sim D$ (cf. Sect. 4.2)
for the most unstable waves so that the geostrophic and
hydrostatic approximation also break down. However, from the
available literature it appears that while the results based on
the model described in Sects. 2 and 3 require some quantitative
refinement when $\Omega \sim N$, they nonetheless predict roughly the
correct length scales and growth rates for the instability. We
summarize this literature briefly.

No general stability conditions are known for the nongeostrophic
or nonhydrostatic case. Nongeostrophic instability has been
studied by Stone (1966, 1970), for an incompressible model. His
results imply that the growth rates and wavelengths predicted
from the geostrophic approximation are quite accurate for this
model. Non-geostrophic, non-hydrostatic instability has been
studied by Stone (1971) and Hyun and Peskin (1976) who showed
that non-hydrostatic conditions somewhat increase the
stability. Geostrophic, hydrostatic instability on a full sphere was studied
by Hollingsworth (1975), Hollingsworth et al. (1976), Simmons
and Hoskins (1976), and Warn (1976). Growth rates quite similar
to those predicted from the $\beta$-plane model were found by all these
authors.

We conclude that in the case of rapid rotation baroclinic
instability may still be studied the framework of the quasi-
geostrophic potential vorticity equation, but with the full effect
of the boundary conditions taken into account. This is beyond the
scope of the present paper.

\subsection{4.2. Slow rotation ($\Omega \ll N$)}

A general qualitative property of an unstable baroclinic wave is that
\[ D \approx \frac{f}{L}, \]  
\hspace{1cm} (21)
where $L$ is the typical horizontal scale (wavelength) of the
instability and $D$ is the vertical extent over which it penetrates.
The interpretation of this relation depends slightly on the
situation considered. If $D$ is identified with the vertical extent of an
incompressible fluid between fixed plates, the value of $L$
obtained
from Eq. (21) is the so-called Eady-cut-off. No instability occurs
for $\lambda > L$ in the Eady model. In stars, if $D$ is taken to be $r$ or
a pressure scale height $H_p$, one finds $L \gg r$ in the case of slow
rotation. Since $\lambda \approx r$ is the largest wavelength that can occur in the
star, we conclude that there is no instability with vertical scales $H_p$
or $r$ (cf. Knobloch and Spruit, 1982). The instability must appear
on smaller vertical scales. In order to satisfy the instability condition (19)
for slow rotation, $q/N^2$ or $\Delta = r \tilde{\omega} \ln(\Omega(r))$ must vary
rapidly enough with $r$. Since the existence of jumps in these
quantities (especially jumps in the molecular weight $\mu$)
cannot be excluded in stellar interiors, it is of interest to study the
instability in such a case.

We consider a model consisting of two semi-infinite layers
which differ in molecular weight but are otherwise the same. Let
\[ M^2 \equiv N^2/\alpha, \]  
and
\[ M = M_+ \quad z > 0 \]  
\[ M = M_- \quad z < 0. \]  
\hspace{1cm} (22)
\hspace{1cm} (23)
and let $\Omega$ and $\alpha$ be constants. This model is quite similar to that
of McIntyre (1972). Our first task is to determine the scales $L$ and $D$
on which the instability can occur. We do this by non-
dimensionalizing the linearized quasi-geostrophic potential
vorticity equation (9) with suitable scales for $x$, $y$, $z$, $u$, and $t$, such
that after ignoring small terms an equation results that still shows
some form of instability. For an extensive discussion and examples
of this procedure, see Pedlosky (1979, Chaps. 6 and 5). This way,
we quickly discover that essential terms in the equation are lost
unless $D$ and $L$ satisfy relation (21). Similarly, the scales $U$
for $u_0$ and $T$ for the time $t$ must satisfy $UT = L$. The appropriate scale
$U$ is the variation in flow speed over a height range $D$. By
comparing Eqs. (1) and (15) we find that a constant value of $\alpha$
corresponds to a linear variation of $u_0$ with $z$, given by
\[ \partial_z u_0 = - \alpha \Omega \cos \lambda. \]  
\hspace{1cm} (24)
Assuming that $\alpha$, $\sin \lambda$, $\cos \lambda$ are all $O(1)$ we may use the following
scales: for
\[ x, y : L, \]  
\hspace{1cm} (25a)
\[ z : D = L \Omega/N_+, \]  
\hspace{1cm} (25b)
\[ u_0 : U = D \Omega, \]  
\hspace{1cm} (25c)
\[ t : T = L/U = N_+ \Omega^2, \]  
\hspace{1cm} (25d)
where $N_+$ is the buoyancy frequency in $z > 0$. Nondimensionalizing Eq. (9)
with respect to these scales, we obtain
\[ (\partial_t - \alpha \cos \lambda \partial_z \partial_z) \left[ \partial_z \phi + \partial_z \phi + \partial_z \phi + \partial_z \phi \right] \left[ \frac{1}{M^2} \partial_z \phi \right] \]  
\[ + 2 \cos \lambda \left( \frac{L N_+}{r \Omega} \right) \left[ 1 + 2 \alpha \sin^2 \lambda \frac{r \Omega}{L N_+} M^2 \partial_z \phi \right] \partial_z \phi = 0. \]  
\hspace{1cm} (26)
where $t$, $x$, $y$, and $z$ are now the dimensionless coordinates. Assuming
that $\alpha$ is of order unity, all terms in the equation are of the
same order of magnitude when
\[ L = r N_+. \]  
\hspace{1cm} (27)
We conclude that the typical horizontal scale of the instability is given by (27), and is therefore much smaller than the stellar radius. The possibility of instability at significantly larger scales, in particular \(L \sim r\), can be excluded by solving (26) by a series expansion in \(L^{-1}\). Together with (25b), relation (27) implies that the geostrophic as well as the hydrostatic and \(\beta\)-plane approximation are relevant in the slowly rotating case \(\Omega/N \ll 1\), so that the use of (26) is justified a posteriori.

The general solution of (26) is rather complicated even for a simple jump in \(M^2\). It is related to the solution for Charney's model (cf. Pedlosky, Chap. 7.8). The essentials can, however, be found by assuming

\[
\varepsilon = \frac{L}{r} \frac{N^2}{\Omega} \ll 1.
\]  

(28)

This procedure gives correct results for wavelengths short compared with the value given by (27). By letting \(\varepsilon \to 1\) afterwards one obtains a reasonable approximation to the behavior of the most unstable wave, which satisfies (27) (cf. McIntyre, 1972).

The solution is obtained by an expansion procedure in \(\varepsilon\). Details are given in Appendix II. The pressure perturbation is, to order zero in \(\varepsilon\):

\[
\phi_0 = \exp(-|z|q),
\]  

(29)

where

\[
q(z) = \frac{1}{2} \frac{1}{|\sin A|} \begin{cases} 1 \quad (z > 0), \\ 1 + \frac{1}{|\sin A|} \frac{M_-}{M_+} \quad (z < 0). \end{cases}
\]

Thus, as expected, the solution decays exponentially far from the interface, on a scale \(D\). If we take the jump in \(\mu\) to be small, so that \(|M_+/M_- - 1| \ll 1\), the dimensionless growth rate is [cf. (A22)]

\[
\sigma \approx 2\pi \cos \alpha \text{ sgn}(\alpha) \frac{M_+ - M_-}{M_+ + M_-}.
\]  

(30)

Inserting \(\varepsilon = 1\), noting that due to our assumptions \(\cos A\) has to be of order unity, say \(\frac{1}{2}\), and transforming to physical units, we find that the dimensional growth rate \(\eta\) is

\[
\eta = \frac{\Omega^2}{N^2} \sigma \approx \frac{\Omega^2}{N^2} \text{ sgn}(\alpha) \frac{M_+ - M_-}{M_+ + M_-}.
\]  

(31)

Equation (31) suggests that the growth rate is independent of \(\alpha\), even when \(\alpha \to 0\). Actually, this limit cannot be taken since the expansion procedure requires that \(\varepsilon \gg \varepsilon\). In reality, the growth rate probably vanishes as \(\alpha \to 0\). It is proportional to the magnitude of the jump, and is therefore at most

\[
\eta \lesssim \frac{\Omega^2}{N},
\]  

(32)

which is a small fraction of the rotation rate. Note that for \(\alpha > 0\) (rotation rate decreasing outwards) instability occurs only when \(M_+ > M_-\). When \(M_+ < M_-\), the potential vorticity gradient [Eq. (18)] is positive definite so that then the differential rotation is stable near the jump.

A jump in \(M^2 = N^2/\Omega\) in general requires the existence of a gradient in molecular weight. Because of the rapid thermal diffusion, a jump in the thermal buoyancy frequency \(N_T\) [cf. Eq. (10)] is possible only for very short periods of time. A jump in \(M^2\) can therefore result in two distinct ways: a) by a jump in \(\mu\), due to a jump in \(\mu\), or b) by a jump in \(N_T^2\), i.e. by a change of slope in the molecular weight distribution \(\mu(r)\).

The case most likely to occur in stellar evolution, i.e. an outward decreasing rotation rate together with a jump-like decrease of \(\mu\), satisfies the condition \(M_+ > M_-\) (unless \(N_T^2\) is considerably less than \(N^2\), so that baroclinic instability is to be expected near such a jump. However, because the scale of the instability in the radial direction is only of the order \(rN^2/\Omega^2\), the instability is a very local phenomenon unless the rotation rate is very high.

Note that a jump in \(N^2/\Omega\), even if it is of the right sign to cause instability, also has to be sufficiently sharp. Condition (19) shows that the thickness \(\Delta r\) of the jump has to satisfy

\[
\Delta r \lesssim r \frac{\Omega^2}{N^2} \Delta \ln M = D \Delta \ln M,
\]  

(33)

where \(D\) is the penetration depth of the instability [cf. Eqs. (25b) and (27)]. For slowly rotating stars, this is a stringent requirement. The most important aspect of the instability at slow rotation is its highly localized nature in the radial direction. Thus the spherical geometry of the star can be disregarded, and the boundary conditions to be applied are unambiguous (the disturbance has to vanish at large distances). This removes the need to consider the star as a whole (in contrast with the rapidly rotating case) and allows simple conclusions about the instability, such as those summarized in Eqs. (21), (27), (31), and (33) to be drawn.

5. Discussion

It appears that the ratio \(\Omega/N\) of rotation rate to buoyancy frequency is the most important parameter for baroclinic instability. When this ratio is of order unity, the instability can affect large portions of a radiatively stratified star. The significance of this finding is reduced somewhat by the fact that ordinary shear instability of the kind discussed by Zahn (1975, 1983) probably sets in at lower values of \(\Omega/N\), especially if the gradient of molecular weight is not too strong (cf. also Spruit et al., 1983). Thus, mixing in rapidly rotating stars may be due mostly to shear instability rather than baroclinic instability.

As \(\Omega/N\) decreases, baroclinic instability becomes restricted to special circumstances. The most interesting of these is a jump in molecular weight. Whereas the other hydrodynamic instabilities are suppressed by such jumps (except probably the ABCD instability), the baroclinic instability has a certain preference for them. It is, however, restricted to a layer of thickness

\[
D \approx r \left(\frac{\Omega}{N}\right)^2
\]  

(34)

near the jump [cf. Eqs. (25b) and (27)], and requires that the jump itself is thinner than \(D\) [cf. Eq. (33)]. Thus, the instability suggests itself as a candidate for mixing across a \(\mu\)-jump, though it will operate only if the rotation is close enough to breakup. The combination of rapid rotation with a \(\mu\)-jump is found in evolutionary models of red giants (Kippenhahn et al., 1970; Mengel and Gross, 1976). This suggests that baroclinic instability is relevant for producing some of the abundance anomalies seen in these stars.

Since \(N \sim r\) near the stellar center, the condition \(\Omega/N \sim 1\) can also be encountered in the center of a slowly rotating star. This

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suggests application to the solar neutrino problem, since mixing of 
H into the center is known to be the simplest way of reducing the 
eutrino flux (Schatzman and Maeder, 1981). The extent in r 
potentially affected can be estimated as follows. Let r_i be the 
radius at which \Omega = N. Then the baroclinic instability affects the 
region r < r_{max}, where 
\[ r_{max} = r_i + D(r_i) \approx 2r_i, \]  
(35)
since D is given by Eq. (34). 
Evaluating N near the center (denoted by subscript c), we obtain 
\[ r_{max} = 2\Omega \left( \frac{4\pi c}{3 G \xi} \right)^{-1} \left[ (F_c - F_s) \frac{\mu_s}{\Omega / R T_s} \right]^{-1/2}, \]  
(36)
where G is the gravitational constant and R the gas constant. As 
an example, for the Sun we find 
\[ r_{max}/R_\odot \approx 3 \times 10^{-6} \frac{\xi}{\Omega_\odot}, \]  
(37)
where \Omega_\odot \approx 3 \times 10^{-6} \text{ s}^{-1} is the surface rotation rate. Evidently, the 
volume affected is negligible for the Sun even when the center 
rotates ten times faster than the surface. 
Due to the small penetration depth D the instability is not of 
more than very local importance in slowly rotating stars unless 
one postulates that a large number of small \mu-jumps exists, 
distributed through a significant portion of the star. This possibility 
is not entirely hypothetical if some form of slow mixing takes 
place in the star, since the interaction of such mixing with a 
stabilizing \mu-gradient can, in principle, produce such jumps (e.g. 
Endal and Sofia, 1981). The number of jumps would have to be of 
the order r/D, so that their amplitude would be 
\[ \frac{\Delta \mu}{\mu} \sim \frac{D}{r} \approx \Omega^2. \] 
In order to get instability, the sharpness of the jumps has to satisfy 
Eq. (33), i.e. \Delta r/r \sim (\Omega/N)^4. As an example, for the solar interior, 
with \Omega/N < 3 \times 10^{-2} (corresponding to \Omega/\Omega_\odot < 10) we have 
\Delta r < 0.4 \text{ kyr}. The timescale for broadening of such sharp jumps, by 
molecular diffusion alone, is \leq 4 \text{ yr}. This is shorter than the 
growth time of the baroclinic instability, \tau \approx N/\Omega^2/\mu \gtrsim 40 \text{ yr}. 
We conclude that baroclinic instability is extremely unlikely to be 
of importance in slowly rotating stars. 
In this paper we have considered only the adiabatic form of 
baroclinic instability. If viscosity and thermal diffusion are present 
additional modes of instability exist, namely the Goldreich-
Schubert-Fricke and ABCD instabilities. Though these also re-
quire the star to be baroclinic, their connection with adiabatic 
baroclinic instability is not direct. This is because they are 
especially nongeostrophic in nature, in sharp contrast with the 
adiabatic instability. Moreover, they depend for their existence on 
the presence of thermal diffusion. 

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Appendix I. Quasigeostrophic potential vorticity equation 
in the case of a variable molecular weight 

We show here that, in the presence of a \mu-gradient, the standard 
form of the potential vorticity equation may be used. We follow the 
analysis given in Pedlosky (1979), but using physical rather than 
dimensionless variables. The vertical vorticity equation [Pedlosky, 
Eq. (6.3.17)] is 
\[ \frac{d}{dt} (\zeta + \beta y) = \frac{f}{\mu_s} \frac{\partial}{\partial z} (q_s w), \]  
(A1)
where 
\[ \frac{d}{dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y}, \]  
(A2)
and the subscripts s, 0 refer respectively to a "reference" stratification 
which depends on z only, and to a small but not infinitesimal deviation from it. The quantity 
\[ \zeta = \frac{\partial}{\partial x} v_0 - \frac{\partial}{\partial y} u_0 \]  
(A3)
is the vertical component of the vorticity, and \beta = (2\Omega/r) \cos \lambda, 
f = 2\Omega \sin \lambda, as before. Equation (A1) is entirely independent of 
the thermal and molecular weight stratification and is valid as long as 
the motions are approximately geostrophic. The potential 
temperature \hat{T} is given by 
\[ \ln \hat{T} = \frac{1}{\gamma} \ln p - \ln \rho \]  
\[ = \left[ \frac{1}{\gamma} \right] \ln p + \ln T - \ln \mu \]  
\[ = \ln T - F_s \ln P - \ln \mu. \]  
(A4)
For adiabatic flow, without exchange of chemical constituents, 
\hat{T} is a conserved quantity so that, with \hat{T}_0, \hat{T}_s \ll 1 and \hat{T}_s \hat{T}_0 \approx \hat{T}_s \hat{T}_0 \approx \hat{T}_s \hat{T}_0:\n\frac{d}{dt} \hat{T}_0 + \omega \frac{\partial}{\partial z} \hat{T}_0 = 0 \]  
(A5)
[cf. Pedlosky, Eq. (6.5.12)]. We define the buoyancy frequency N 
through 
\[ \frac{1}{\gamma} \frac{\partial}{\partial z} (q_s w) = - \frac{\beta}{\mu_s} \frac{\partial}{\partial z} \left( \frac{q_s}{N^2} \frac{\partial}{\partial z} \hat{T}_0 \right) \]  
\[ + \frac{1}{N^2 \hat{T}_0} \left( \frac{\partial}{\partial z} u_0 \frac{\partial}{\partial z} \hat{T}_0 + \frac{\partial}{\partial y} v_0 \frac{\partial}{\partial y} \hat{T}_0 \right), \]  
(A7)
[cf. Pedlosky, Eq. (6.5.15); note that there is a sign error in the last 
line of his equation]. The hydrostatic approximation yields 
\[ \frac{\partial}{\partial z} p_0 = - \frac{\beta}{\partial z} \frac{p_0}{p_s} + \frac{\mu_0}{\mu_s} \frac{T_0}{T_s} \]  
\[ = - \frac{\beta}{\partial z} \left( \frac{T_0}{T_s} + \frac{p_0}{\gamma p_s} \right) \]  
\[ \approx \frac{\beta}{\partial z} \frac{T_0}{T_s}, \]  
(A8)
since the vertical scale D on which p_0 varies is small compared 
with the pressure scale height p_s/\beta.
Together with the geostrophic approximation [Eq. (7)], Eq. (A8) yields the thermal wind relation

\[ f\phi_0 - \frac{g}{T_x} \frac{\partial}{\partial z} \left( \frac{\partial T_0}{\partial z} \right) = 0, \tag{A9a} \]
\[ f\phi_0 - \frac{g}{T_x} \frac{\partial}{\partial z} \left( \frac{\partial T_0}{\partial z} \right) = 0, \tag{A9b} \]
so that the last term in (A7) vanishes, and the quasigeostrophic potential vorticity equation becomes

\[ \frac{d}{dt} \left[ \zeta_0 + \beta y + \frac{g}{\theta_x} \frac{\partial}{\partial z} \left( \frac{\partial \phi_0}{\partial z} \right) \right] = 0, \tag{A10} \]

or with (A8):

\[ \frac{d}{dt} \left[ \zeta_0 + \beta y + \frac{f}{\theta_x} \frac{\partial}{\partial z} \left( \frac{\partial \phi_0}{\partial z} \right) \right] = 0. \tag{A11} \]

Linearization of this equation yields Eq. (9), if a mean flow with \( \nu_0 = 0 \) is assumed (corresponding to pure differential rotation), and the pressure fluctuation \( p \) is denoted by \( \phi \).

Appendix II. Instability near a discontinuity

in the molecular weight

In this appendix we demonstrate that a jump in the molecular weight \( \mu \) can precipitate a baroclinic instability. Our discussion follows closely a similar calculation given by McIntyre (1972). We solve Eq. (26) for the case (23) under the assumption (28), and seek solutions of the form \( \phi(z) \exp(ikx + ily - \omega t) \) subject to the boundary condition \( \phi(z) \rightarrow 0 \) as \( z \rightarrow \pm \infty \), and appropriate jump conditions across \( z = 0 \). Since the scale \( L \) was taken to correspond with the horizontal wavelength of the instability, we have \( k^2 + l^2 = 1 \). Equation (26) can then be written as

\[ -(\lambda z + c_0) \left[ -\phi + a^2 M^2 \frac{\partial}{\partial z} \left( \frac{1}{M^2} \frac{\partial \phi}{\partial z} \right) \right] + [2\alpha \cos \lambda + b\delta(z)] \phi = 0, \tag{A12} \]

where \( c = \omega / k \) is the phase speed, and

\[ \lambda = a \cos \lambda, \quad a^2 = 4 \sin^2 \lambda, \quad b = 4 \lambda \sin^2 \lambda (1 - M^2 / M^2). \tag{A13} \]

We expand the solution in a power series

\[ \phi = \phi_0 + \epsilon \phi_1 + \ldots \tag{A14} \]

To zeroth order \( \phi_0 \) solves

\[ -(\lambda z + c_0) \left[ -\phi_0 + a^2 M^2 \frac{\partial}{\partial z} \left( \frac{1}{M^2} \frac{\partial \phi_0}{\partial z} \right) \right] + b\delta(z) \phi_0 = 0. \tag{A15} \]

With the boundary conditions \( \phi_0 \rightarrow 0 \) as \( z \rightarrow \pm \infty \) and \( [\phi_0] \rightarrow 0 \), the solution is

\[ \phi_0 = \exp(-q|z|), \tag{A16} \]

where

\[ q(z) = \frac{1}{2} \sin |\lambda|^{-1} \quad z > 0 \tag{A17} \]
\[ = \frac{1}{2} \sin |\lambda|^{-1} M_- / M_+ \quad z < 0. \tag{A17} \]

The phase velocity \( c_0 \) is determined by dividing (A15) by \( \lambda z + c_0 \) and integrating across the jump at \( z = 0 \). We obtain

\[ c_0 = \left( \frac{M_+}{M_-} - 1 \right) \alpha \sin 2\lambda. \tag{A18} \]

Note that, unlike in McIntyre's case, the wave speed is proportional to the jump in \( M \). The difference is due to the fact that in his case there is a jump in \( z \) as well as one in \( M \).

At \( \theta(\theta) \) we obtain

\[ -(\lambda z + c_0) \left[ -\phi_1 + a^2 M^2 \frac{\partial}{\partial z} \left( \frac{1}{M^2} \frac{\partial \phi_1}{\partial z} \right) \right] + b\delta(z) \phi_1 \]
\[ = c_1 \delta(z) \phi_0 / \lambda \quad \sin \phi_0 / \lambda > 0. \tag{A19} \]

The solvability condition for \( \phi_1 \) yields (cf. McIntyre, 1972)

\[ c_1 \int_{-\infty}^{\infty} \phi_0 \phi_1 dz = \int_{-\infty}^{\infty} 2 \cos \lambda \phi_0 dz = 0, \tag{A20} \]

so that

\[ c_1 = \pm 2 \pi i \left( \frac{\cos^2 |\lambda|^{-1}}{\sin |\lambda|^{-1}} \right) \sin \lambda \phi_0 > 0. \tag{A21} \]

The sign depends on the contour taken in evaluating the second integral, and must be chosen such that \( c_1 < 0 \) when the situation is stable, i.e., when the coefficient of \( \phi \) in the second term of (A12) is positive definite [cf. condition (19)]. This is the case when \( b > 0 \), i.e., \( \text{sgn}(\lambda)(M_+ - M_-) < 0 \). Equation (A21) thus yields

\[ c_1 = 2 \pi i \cos \lambda \sin \lambda \phi_0 (M_+ - M_-) = (2 - 1)(M_+ - M_-) M_+ < M_+ \]
\[ M_+ - M_- > 1 \quad (2 - 1)(M_+ - M_-) < 0. \tag{A22} \]

The growth rate of the instability, \( \eta = (Q^2 N^2) \sin(2\lambda) \), is therefore positive if \( M_+ > M_- (\lambda > 0) \). Note that \( \epsilon c_1 \leq c_0 \) only if \( \pi / 2 \leq \lambda \), so that the expansion procedure is not valid uniformly for small \( \lambda \).

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